TRANSACTIONS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XXV. PART I.—FOR THE SESSION 1867-68.

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OF

EDINBURGH.

VOL. XXV.

EDINBURGH:
PUBLISHED BY ROBERT GRANT & SON, 54 PRINCES STREET.
AND WILLIAMS & NORGATE, 14 HENRIETTA STREET, COVENT GARDEN, LONDON.

MDCCCLXIX.

BRITISH 7 MA 70 MUSEUM

ROYAL SOCIETY OF EDINBURGH.

THE KEITH, BRISBANE, AND NEILL PRIZES.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The KEITH PRIZE, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1869–70, for the "best communication on a scientific subject, communicated, in the first instance, to the Royal Society during the Sessions 1867–68 and 1868–69." Preference will be given to a paper containing a discovery.

II. MAKDOUGALL BRISBANE PRIZE.

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded at the commencement of the Session 1870–71, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 1st June 1870.

3. The competition is open to all men of science.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society. They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the Paper shall be published in the Transactions.

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III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr PATRICK NEILL of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate,

1. The NEILL PRIZE, consisting of a Gold Medal and a sum of Money, will be awarded at the commencement of the Session 1871–72.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented to the Society during the three years preceding the 1st May 1871,—or failing presentation of a Paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award.

AWARDS OF THE KEITH, MAKDOUGALL BRISBANE, AND NEILL PRIZES, SINCE 1865.

AWARD OF THE KEITH PRIZE.

20th Biennial Period, 1865-67. Professor C. Piazzi Smyth, for his paper on "Recent Measures at the Great Pyramid," published in the Transactions of the Society.

MAKDOUGALL BRISBANE PRIZE.

5th Biennial Period, 1866-68. Dr Alex Crum Brown and Dr Thomas Richard Fraser, for their conjoint paper on "The Connection between Chemical Constitution and Physiological Action," published in the Transactions of the Society.

AWARD OF THE NEILL PRIZE.

4th Triennial Period, 1865-68. Dr William Carmichael M'Intosh, for his paper on "The Structure of the British Nemerteans, and on some New British Annelids," to be published in the Transactions.

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AWARD OF THE KEITH PRIZE.

21ST BIENNIAL PERIOD, 1867–69.	Professor PETER GUTHRIE TAIT, for his paper "On the Rotation of a Rigid Body about a Fixed Point," published in the Transactions of the Society.

LAWS

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AS REVISED 31st OCTOBER 1869.

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The fees of Ordinary Fellows residing in Scotland.

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All Fellows who shall have paid Twenty-five years' annual contribution shall be exempted from farther payment.

Payment to cease after 25 years.

IV.

The fees of admission of an Ordinary Non-Resident Fellow shall be £26, 5s., payable on his admission; and in case of any Non-Resident Fellow coming to reside at any time in Scotland, he shall, during each year of his residence, pay the usual annual contribution of £3, 3s., payable by each Resident Fellow; but after payment of such annual contribution for eight years, he shall be exempt from any farther payment. In the case of any Resident Fellow ceasing to reside in Scot-

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Case of Fellows becoming Non-Resident.

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* At the Meeting of the Society, on the 5th January 1857, when the reduction of the Contributions from £3, 3s., to £2, 2s., from the 11th to the 25th year of membership, was adopted, it was resolved that the existing Members shall share in this reduction, so far as regards their future annual Contributions.

A modification of this rule, in certain cases, was agreed to 3d January 1831.

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V.

Defaulters.

Members failing to pay their contributions for three successive years (due application having been made to them by the Treasurer) shall be reported to the Council, and, if they see fit, shall be declared from that period to be no longer Fellows, and the legal means for recovering such arrears shall be employed.

VI.

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None but Ordinary Fellows shall bear any office in the Society, or vote in the choice of Fellows or Office-Bearers, or interfere in the patrimonial interests of the Society.

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The number of Ordinary Fellows shall be unlimited.

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The Ordinary Fellows, upon producing an order from the TREASURER, shall be entitled to receive from the Publisher, gratis, the Parts of the Society's Transactions which shall be published subsequent to their admission.

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No person shall be proposed as an Ordinary Fellow without a recommendation subscribed by *One* Ordinary Fellow, to the purport below.* This recommendation shall be delivered to the Secretary, and by him laid before the Council, and shall afterwards be printed in the circulars for three Ordinary Meetings of the Society, previous to the day of the election, and shall lie upon the table during that time.

X.

Honorary Fellows, British and Foreign.

Honorary Fellows shall not be subject to any contribution. This class shall consist of persons eminently distinguished for science or literature. Its number shall not exceed Fifty-six, of whom Twenty may be British subjects, and Thirty-six may be subjects of foreign states.

---

* " A. B., a gentleman well skilled in several branches of Science (*or Polite Literature, as the case* " *may be*), being to my knowledge desirous of becoming a Fellow of the Royal Society of Edin- " burgh, I hereby recommend him as deserving of that honour, and as likely to prove a useful and " valuable Member."

This recommendation to be accompanied by a request of admission signed by the Candidate.

XI.

Royal Personages.

Personages of Royal Blood may be elected Honorary Fellows, without regard to the limitation of numbers specified in Law X.

XII.

Recommendation of Honorary Fellows.

Honorary Fellows may be proposed by the Council, or by a recommendation (in the form given below*) subscribed by three Ordinary Fellows; and in case the Council shall decline to bring this recommendation before the Society, it shall be competent for the proposers to bring the same before a General Meeting. The election shall be by ballot, after the proposal has been communicated *viva voce* from the Chair at one meeting, and printed in the circulars for two ordinary meetings of the Society, previous to the day of election.

Mode of Election.

XIII.

Election of Ordinary Fellows.

The election of Ordinary Fellows shall take place at the Ordinary Meetings of the Society. The election shall be by ballot, and shall be determined by a majority of at least two-thirds of the votes, provided Twenty-four Fellows be present and vote.

XIV.

Ordinary Meetings.

The Ordinary Meetings shall be held on the first and third Mondays of every month from November to June inclusive. Regular Minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make.

XV.

The Transactions.

The Society shall from time to time publish its Transactions and Proceedings. For this purpose the Council shall select and arrange the papers which they shall deem it expedient to publish in the *Transactions* of the Society, and shall superintend the printing of the same.

---

* We hereby recommend ______________________________
for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to (*Literature or Science, as the case may be*) believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

____
____
____

To the President and Council of Royal Society
of Edinburgh.

XVI.

How Published.

The Transactions shall be published in Parts or *Fasciculi* at the close of each Session, and the expense shall be defrayed by the Society.

The Council.

There shall be elected annually, for conducting the publications and regulating the private business of the Society, a Council, consisting of a President; Six Vice-Presidents, two at least of whom shall be resident; Twelve Councillors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator of the Museum and Library.

XVII.

Retiring Councillors.

Four Councillors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XVIII.

Election of Office-Bearers.

An Extraordinary Meeting for the Election of Office-Bearers shall be held on the fourth Monday of November annually.

XIX.

Special Meetings: how called.

Special Meetings of the Society may be called by the Secretary, by direction of the Council; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such Meetings.

XX.

Treasurer's Duties.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the Extraordinary Meeting in November, he shall present the accounts for the preceding year, duly audited. At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXI.

Auditor.

At the Extraordinary Meeting in November, a professional accountant shall be chosen to audit the Treasurer's accounts for that year, and to give the necessary discharge of his intromissions.

XXII.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes, he shall, when necessary, employ a clerk, to be paid by the Society. General Secretary's Duties.

The Secretaries to the Ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered; they shall specify all the Donations received, and furnish a list of them, and of the donors' names, to the Curator of the Library and Museum: they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty. Secretaries to Ordinary Meetings.

XXIII.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the Hall, for the inspection of the Fellows. Curator of Museum and Library.

XXIV.

All Articles of the above description shall be open to the inspection of the Fellows at the Hall of the Society, at such times and under such regulations, as the Council from time to time shall appoint. Use of Museum and Library.

XXV.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date. Register Book.

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ERRATA IN LIST OF MEMBERS.

The following Name to be omitted.

JAMES ANSTRUTHER, Esq., died 1866.

The following Name to be inserted.

1868. J. SAMPSON GAMGEE, Esq., Surgeon to the Queen's Hospital, Birmingham.

ERRATA.

In Professor TAIT'S Paper,

Page 281, line 6, omit the sign of integration.

In Dr DICKSON'S Paper,

Page 641, note, line 1, omit "only."
Page 646, line 3, *for* "extremities" *read* extremity."
Plate XXIX. fig. 14, *for* "pt" *read* "pl."

LAWS

OF THE

ROYAL SOCIETY OF EDINBURGH,

AS REVISED 31ST OCTOBER 1866.

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Case of Fellows becoming Non-Resident.

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XIV.

Ordinary Meetings.

The Ordinary Meetings shall be held on the first and third Mondays of every month from November to June inclusive. Regular Minutes shall be kept of the proceedings, and the Secretaries shall do the duty alternately, or according to such agreement as they may find it convenient to make.

XV.

The Transactions

The Society shall from time to time publish its Transactions and Proceedings. For this purpose the Council shall select and arrange the papers which they shall deem it expedient to publish in the *Transactions* of the Society, and shall superintend the printing of the same.

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* We hereby recommend ______________________________
for the distinction of being made an Honorary Fellow of this Society, declaring that each of us from our own knowledge of his services to (*Literature or Science, as the case may be*) believe him to be worthy of that honour.

(To be signed by three Ordinary Fellows.)

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To the President and Council of Royal Society
of Edinburgh.

XVI.

How Published.

The Transactions shall be published in Parts or *Fasciculi* at the close of each Session, and the expense shall be defrayed by the Society.

The Council.

There shall be elected annually, for conducting the publications and regulating the private business of the Society, a Council, consisting of a President; Six Vice-Presidents, two at least of whom shall be resident; Twelve Councillors, a General Secretary, Two Secretaries to the Ordinary Meetings, a Treasurer, and a Curator of the Museum and Library.

XVII.

Retiring Councillors.

Four Councillors shall go out annually, to be taken according to the order in which they stand on the list of the Council.

XVIII.

Election of Office-Bearers.

An Extraordinary Meeting for the Election of Office-Bearers shall be held on the fourth Monday of November annually.

XIX.

Special Meetings; how called.

Special Meetings of the Society may be called by the Secretary, by direction of the Council; or on a requisition signed by six or more Ordinary Fellows. Notice of not less than two days must be given of such Meetings.

XX.

Treasurer's Duties.

The Treasurer shall receive and disburse the money belonging to the Society, granting the necessary receipts, and collecting the money when due.

He shall keep regular accounts of all the cash received and expended, which shall be made up and balanced annually; and at the Extraordinary Meeting in November, he shall present the accounts for the preceding year, duly audited. At this Meeting, the Treasurer shall also lay before the Council a list of all arrears due above two years, and the Council shall thereupon give such directions as they may deem necessary for recovery thereof.

XXI.

Auditor.

At the Extraordinary Meeting in November, a professional accountant shall be chosen to audit the Treasurer's accounts for that year, and to give the necessary discharge of his intromissions.

XXII.

The General Secretary shall keep Minutes of the Extraordinary Meetings of the Society, and of the Meetings of the Council, in two distinct books. He shall, under the direction of the Council, conduct the correspondence of the Society, and superintend its publications. For these purposes, he shall, when necessary, employ a clerk, to be paid by the Society.

General Secretary's Duties.

The Secretaries to the Ordinary Meetings shall keep a regular Minute-book, in which a full account of the proceedings of these Meetings shall be entered; they shall specify all the Donations received, and furnish a list of them, and of the donors' names, to the Curator of the Library and Museum: they shall likewise furnish the Treasurer with notes of all admissions of Ordinary Fellows. They shall assist the General Secretary in superintending the publications, and in his absence shall take his duty.

Secretaries to Ordinary Meetings.

XXIII.

The Curator of the Museum and Library shall have the custody and charge of all the Books, Manuscripts, objects of Natural History, Scientific Productions, and other articles of a similar description belonging to the Society; he shall take an account of these when received, and keep a regular catalogue of the whole, which shall lie in the Hall, for the inspection of the Fellows.

Curator of Museum and Library.

XXIV.

All Articles of the above description shall be open to the inspection of the Fellows at the Hall of the Society, at such times and under such regulations, as the Council from time to time shall appoint.

Use of Museum and Library.

XXV.

A Register shall be kept, in which the names of the Fellows shall be enrolled at their admission, with the date.

Register Book

LIST OF THE ORDINARY FELLOWS OF THE SOCIETY.

N.B.—*Those marked * are Annual Contributors.*

1846 *Alex. J. Adie, Esq., Rockville, Linlithgow
1868 *Rev. Dr David Aitken, 4 Charlotte Square
1866 *Col. Sir James E. Alexander of Westerton
1867 *Rev. Dr W. Lindsay Alexander, 17 Brown Square
1848 Dr James Allan, Inspector of Hospitals, Portsmouth
1856 *Dr G. J. Allman (SECRETARY), Professor of Natural History, 21 Manor Place
1849 *David Anderson, Esq., Moredun, Edinburgh
1845 *Dr Thomas Anderson, Prof. Chemistry, Univ., Glasgow
1823 Warren Hastings Anderson, Esq., Isle of Wight
1867 *Thomas Annandale, Esq., 34 Charlotte Square 10
1840 James Anstruther, Esq., W.S.
1862 *T. C. Archer, Esq., Director of the Museum of Science and Art, 9 Argyle Square
1849 *His Grace the Duke of Argyll (HON. VICE-PRESIDENT), Inverary Castle
1820 Charles Babbage, K.H., London
1843 David Balfour, Esq., Trenaby
1835 Dr J. H. Balfour (GENERAL SECRETARY), Professor of Medicine and Botany, 27 Inverleith Row
1867 *George F. Barbour, Esq., 11 George Square
1862 *Hon. Lord Barcaple, 3 Ainslie Place
1830 Dr Thomas Barnes, Carlisle
1858 Edmund Chisholm Batten, M.A., Lincoln's Inn, London 20
1844 Dr Begbie, 10 Charlotte Square
1843 Dr Bennett, Professor of Institutes of Medicine, 1 Glenfinlas Street
1861 *George Berry, Esq., 2 Windsor Terrace, Portobello
1866 *Adam Black, Esq., 38 Drummond Place
1850 *Hugh Blackburn, Esq., Prof. Mathematics, University, Glasgow.
1863 *Professor Blackie, 24 Hill Street
1857 *John Blackwood, Esq., 3 Randolph Crescent
1862 *Rev. Dr W. G. Blaikie, New College
1854 Ernest Bonar, Esq.
1863 *William Brand, Esq., 5 Northumberland Street 30
1864 *Dr Alex. Crum Brown, 4 Rilibank Terrace
1859 *Dr John Brown, 23 Rutland Street
1861 *Rev. Thomas Brown, 16 Carlton Street
1835 William Brown, Esq., 25 Dublin Street
1861 *W. A. F. Browne, Esq., Post-Office Buildings
1867 *A. H. Bryce, D.C.L., LL.D., 42 Moray Place.
1856 *David Bryce, Esq., Architect, 131 George Street
1833 His Grace the Duke of Buccleuch, K.G., Dalkeith Palace
1857 *Dr W. M. Buchanan, 3 Carlton Terrace
1847 *J. H. Burton, LL.D., Advocate, Craig House 40
1865 *Alfred R. Catton, B.A., College
1866 *David Chalmers, Esq., Kate's Mill, Slateford
1840 Robert Chambers, LL.D., St Andrews
1860 *William Chambers, Esq. of Glenormiston, 13 Chester Street
1823 Dr Christison, Professor of Materia Medica (VICE-PRESIDENT), 40 Moray Place
1863 Dr H. F. C. Cleghorn, Stravithy, St Andrews
1856 *Thomas Cleghorn, Esq., Advocate, 26 Queen Street
1844 Dr Thomas R. Colledge, Lauriston House, Cheltenham
1829 The Right Honourable Lord Colonsay, London
1829 A. Colyar, Esq. 50
1850 *Dr James Scarth Combe, 36 York Place
1866 *Thomas Constable, Esq., 11 Thistle Street
1843 Dr John Rose Cormack, Orleans, France
1843 Andrew Coventry, Esq., Advocate, 29 Moray Place
1863 *Charles Cowan, Esq., Valleyfield, Penicuik
1854 *Sir James Coxe, M.D., Kinellan
1830 J. T. Gibson-Craig, Esq., W.S., 24 York Place
1829 Sir William Gibson-Craig, Bart., Riccarton
1853 Rev. John Cumming, D.D., London
1852 *James Cunningham, Esq., W.S., 50 Queen Street 60
1823 Liscombe J. Curtis, Esq., Ingsdown House, Devonshire
1851 *E. W. Dallas, Esq., 125 Princes Street
1841 James Dalmahoy, Esq., 9 Forres Street
1852 *Allen Dalzell, M.D., The Lodge, North Berwick
1862 *Nicholas Alexander Dalzell, Esq., Conservator of Forests, Bombay
1868 *Dr Robert Daun, 6 Picardy Place
1867 *David Davidson, Esq., Bank of Scotland
1848 *Henry Davidson, Esq., Muirhouse
1867 Henry Dircks, Esq., C.E., London
1867 *Francis Deas, Esq., LL.B., Advocate, 32 Heriot Row 70
1863 *W. Dittmar, Esq., College
1867 *James Donaldson, Esq., LL.D., 8 Mayfield Street
1866 *David Douglas, Esq., 11 Salisbury Road
1839 Francis Brown Douglas, Esq., Advocate, 21 Moray Pl.
1868 *Rev. D. T. K. Drummond, B.A., 6 Montpelier
1867 *G. Stirling Home Drummond, Esq, Blair-Drummond
1860 *Patrick Dudgeon, Esq. of Cargen
1863 *Dr J. Matthews Duncan, 30 Charlotte Square
1851 *Sir David Dundas, Bart. of Dunira
1859 *Rev. Dr John Duns, 4 Mansion-House Road, Grange 80
1866 *Dr James Dunsmure, 53 Queen Street
1864 *Dr Robert Dyce, Prof. of Midwifery, Aberdeen
1856 *W. Mitchell Ellis, Esq., Wellington Lodge, Portobello
1855 Robert Etheridge, Esq., Clifton, Bristol
1866 *William Euing, Esq., Glasgow
1863 *J. D. Everett, LL.D., Prof. Nat. Phil., Queen's College, Belfast
1866 *James Falshaw, Esq., C.E., 26 Castle Street
1859 *Dr Fayrer, Professor of Surgery, Calcutta
1868 *Dr Robert M. Ferguson, 3 Rillbank Crescent
1858 Frederick Field, Esq., Chili 90
1852 Dr Andrew Fleming, H.M.I.S., Bengal

1831 Principal Forbes (VICE-PRESIDENT), St Andrews
1859 Major James George Forlong, Bombay
1828 John Forster, Esq., Liverpool
1864 *Dr John Foulerton, Manila
1858 *Professor Fraser, 20 Chester Street
1867 *Dr Thomas R. Fraser, College
1867 *Frederick Fuller, Esq., Prof. Math., Univ., Aberdeen
1867 Dr Charles Gainer, Oxford
1867 *Dr Arthur Gamgee, 27 Alva Street 100
1861 *Archibald Geikie, Esq., 16 Duncan Street, Newington
1868 *Rev. Joseph Taylor Goodsir, 11 Danube Street
1845 *L. D. B. Gordon, Esq., C.E., London
1850 *Lieut.-Col. W. D. Gosset, R.E.
1867 *Dr Andrew Graham, R.N., 35 Melville Street
1851 *Rev. Dr James Grant, D.C.L., 18 Great King Street
1824 Dr Robert E. Grant, Prof. Comp. Anat., Univ. Coll., London
1860 *Dr Frederick Guthrie, M.A., Prof. of Chemistry, Roy. Coll., Mauritius
1868 *Col. Seton Guthrie, Thurso
1868 *Rev. Dr Thomas Guthrie, 1 Salisbury Road 110
1867 *Dr D. R. Haldane, 22 Charlotte Square
1867 *Frederick Hallard, Esq., Advocate, 7 Whitehouse Terrace
1867 *James H. B. Hallen, Esq., Canada
1833 Alexander Hamilton, LL.B., W.S., The Elms, Whitehouse Loan
1837 Dr P. D. Handyside, 11 Hope Street
1864 *Rev. Dr Hannah, Glenalmond
1854 Professor Robert Harkness, Queen's College, Cork
1867 *Sir George Harvey, 21 Regent Terrace
1859 *G. W. Hay, Esq. of Whiterigg
1855 *James Hay, Esq., 5 Links Place, Leith 120
1862 *Dr James Hector, New Zealand
1859 Lieut. John Hills, Bombay Engineers
1828 David Milne Home, Esq. of Wedderburn (VICE-PRESIDENT), 10 York Place
1839 Dr Adam Hunter, 18 Abercromby Place
1864 *Robert Hutchison, Esq., Carlowrie Castle
1855 *The Right Hon. John Inglis, Lord Justice-General, 30 Abercromby Place
1858 *Professor Innes (VICE-PRESIDENT), Inverleith House
1840 Edward J. Jackson, Esq., 6 Coates Crescent
1863 William Jameson, Esq., Surgeon-Major, Saharunpore
1860 *George A. Jamieson, Esq., 58 Melville Street 130
1825 Sir William Jardine, Bart., LL.D., of Applegarth, Jardine Hall, Lockerby
1865 *Charles Jenner, Esq., Easter Duddingston Lodge
1863 *Hon. Charles Baillie, LL.D., Lord Jerviswoode, 10 Strathearn Road
1850 *Alex. K. Johnston, LL.D., March-Hall Park, Prestonfield
1867 *T. B. Johnston, Esq., 9 Claremont Crescent
1867 *William Keddie, Esq., Glasgow
1866 *Dr Alexander Keiller, 21 Queen Street
1839 Rev. Prof. Kelland (VICE-PRESIDENT), 20 Clarendon Crescent
1868 *Thomas Key, Esq., 42 George Square
1868 *J. W. Laidlay, Esq., Seacliff 140
1863 *Charles Lawson, Esq., 35 George Square
1865 *Charles Lawson, jun., Esq., 34 George Square
1856 *Dr Laycock, Professor of the Practice of Medicine, 13 Walker Street
1863 *Hon. G. Waldegrave Leslie, 4 Heriot Row
1858 *James Leslie, Esq., C.E., 2 Charlotte Square
1861 *Dr W. Lauder Lindsay, Gilgal, Perth
1864 *William Lindsay, Esq., Hermitage-Hill House, Leith
1857 Thomas Login, Esq., C.E., Pegu
1861 *Professor Lorimer, Advocate, 21 Hill Street
1849 *Dr W. H. Lowe, Balgreen, Slateford 150
1855 *Dr Stevenson Macadam, 25 Brighton Place, Portobello
1861 *Dr James M'Bain, R.N., Logie Villa, York Road, Trinity
1868 *Dr Thomas Smith Maccall, Polmont
1867 *John M. M'Candlish, Esq., 4 Doune Terrace
1866 *John M'Culloch, Esq., Banker, 11 Duke Street
1820 Dr Wm. Macdonald, Prof. Civ. and Nat. Hist., St Andrews
1847 *W. Macdonald Macdonald, Esq., St Martins
1861 *Dr A. E. Mackay, R.N.
1840 John Mackenzie, Esq., 11 Abercromby Place
1843 Dr A. Douglas Maclagan (CURATOR), Prof. of Medical Jurisprudence, 28 Heriot Row 160
1853 Lieut.-Col. R. Maclagan, Royal Engineers, Bengal
1864 *Peter M'Lagan, Esq. of Pumpherston, M.P.
1868 *John F. M'Lennan, Esq., Advocate, 22 Hill Street
1868 *John Macmillan, Esq., M.A., 16 Buccleuch Place
1866 *John Macnair, Esq., 33 Moray Place
1840 Sir John M'Neill, G.C.B., Granton House
1858 *Dr R. B. Malcolm, 126 George Street
1864 *J. D. Marwick, Esq., 10 Bellevue Crescent
1866 *Professor David Masson, 3 Rosebery Crescent
1856 *James Clerk Maxwell, Esq., late Prof. Nat. Phil., King's College, London, Glenlair, Kirkpatrick-Durham 170
1849 *Sir William Stirling-Maxwell, Bart., Keir, M.P.
1863 *Edward Meldrum, Esq., Bathgate
1853 *Græme Reid Mercer, Esq., Ceylon Civil Service
1841 John Miller, Esq., C.E., 2 Melville Crescent
1818 Dr P. Miller, Exeter
1852 *Thomas Miller, Esq., A.M., LL.D., Rector, Perth Academy
1833 Rear-Admiral Sir Alexander Milne, R.N., Inveresk
1866 *Dr Arthur Mitchell, 6 Laverock Bank Villas
1843 Joseph Mitchell, Esq., C.E., Inverness
1865 *Dr John Moir, 52 Castle Street 180
1868 *Rev. James F. Montgomery, 7 Walker Street
1866 *Dr Charles Morehead, 1 Clarendon Crescent
1866 *Right Rev. Bishop Morrell, Bishop's Court, Bruntsfield Place
1861 *John Muir, D.C.L., LL.D., 16 Regent Terrace
1824 Rev. Dr William Muir, Ormelie Villa, Murrayfield
1857 Dr John Ivor Murray, Colonial Surgeon, Hong Kong
1850 Dr Sheridan Muspratt, Liverpool
1842 Robert Nasmyth, Esq., 5 Charlotte Square
1856 *Hon. Lord Neaves, 7 Charlotte Square
1866 *Thomas Nelson, Esq., Abden House, Prestonfield 190
1847 *James Nicol, Esq., Prof. Nat. Hist., Aberdeen
1863 *Hon. Lord Ormidale, 14 Moray Place
1863 *David Page, LL.D., 44 Gilmore Place
1837 Dr Richard Parnell, Melrose
1863 *Dr Alexander Peddie, 15 Rutland Street

1868 *John Dick Peddie, Esq., Architect, 33 Buckingham Terrace
1856 *Dr Penny, Glasgow
1849 *W. Pirrie, Esq., Prof. Surg., Marischal Coll., Aberdeen
1859 *Lyon Playfair, C.B., LL.D. (VICE-PRESIDENT), Prof. Chemistry, 14 Abercromby Place
1834 Mungo Ponton, Esq., W.S., Clifton, Bristol 200
1852 Eyre B. Powell, Esq., Director of Public Instruction, Madras
1865 *James Powrie, Esq., Reswallie, Forfar
1849 *Hon. B. F. Primrose, 22 Moray Place
1868 *Samuel Raleigh, Esq., 30 George Square
1827 Very Rev. E. B. Ramsay, LL.D., 23 Ainslie Place
1850 *W. J. M. Rankine, Esq., C.E., Prof. Civil Engineering, University, Glasgow
1865 *Rev. Francis Redford, M.A., Silloth
1836 David Rhind, Esq., Architect, 54 Great King Street
1818 William Richardson, Esq., Cheltenham
1840 Martyn J. Roberts, Esq., Crickhowell, South Wales 210
1859 *George Robertson, Esq., C.E., 47 Albany Street
1832 Dr Montgomery Robertson, Mortlake, Surrey
1860 *Dr William Robertson, 28 Albany Street
1862 *Dr E. Ronalds, Bonnington Road
1852 *Alex. James Russell, Esq., C.S., 9 Shandwick Place
1837 J. Scott Russell, Esq., 5 Westminster Chambers, London
1859 *Robert Russell, Esq., Pilmuir, Leven, Fife
1863 *James Sanderson, Esq., Surgeon-Major, 17 Claremont Crescent
1864 *Rev. D. F. Sandford, 19 Rutland Street
1849 *Edward Sang, Esq., 2 George Street 220
1846 *Dr Schmitz, London
1853 *Hugh Scott, Esq. of Gala, Galashiels
1840 Sir William Scott, Bart., Ancrum
1864 *Professor Sellar, 15 Buckingham Terrace
1850 *Dr William Seller, 18 Northumberland Street
1834 Dr Sharpey, Prof. Anatomy, Univ. Coll., London
1844 Sir James Y. Simpson, Bart., Prof. of Midwifery, 52 Queen Street
1829 Ven. Archdeacon Sinclair, Kensington
1859 *William F. Skene, LL.D., W.S., 20 Inverleith Row
1868 *Adam Gillies Smith, Esq., C.A., 5 Lennox Street 230
1837 Arch. Smith, Esq., Lincoln's Inn, London
1839 David Smith, Esq., W.S. (TREASURER), 10 Eton Terrace
1863 *Dr John Alex. Smith, 7 West Maitland Street
1866 *Dr John Smith, 20 Charlotte Square
1855 *R. M. Smith, Esq., 4 Bellevue Crescent
1846 *Professor Piazzi Smyth, 1 Hillside Crescent
1866 *Professor Spence, 21 Ainslie Place
1850 *Dr James Stark, 21 Rutland Street
1843 Henry Stephens, Esq., Red Braes Cottage, Bonnington
1847 *Moses Steven, Esq. of Bellahouston, 12 Manor Place 240
1844 David Stevenson, Esq., C.E., 25 Royal Terrace
1868 *John J. Stevenson, Esq., Glasgow
1848 *Thomas Stevenson, Esq., C.E., 17 Heriot Row
1858 *Rev. Dr Stevenson, Prof. Eccl. Hist., 37 Royal Terrace
1868 *Major J. H. M. Shaw Stewart, Royal Engineers, Madras
1866 *Dr T. Grainger Stewart, 25 Queen Street
1848 *Patrick James Stirling, Esq., LL.D., Kippendavie House
1823 Captain T. D. Stuart, H.M.I.S.
1848 *William Swan, Esq., Professor of Natural Philosophy, St Andrews
1844 Archibald Campbell Swinton, Esq., Kimmerghame 250
1830 Professor Syme, Millbank House, Canaan
1854 Dr John Addington Symonds, Clifton, Bristol
1861 *Professor P. Guthrie Tait (SECRETARY), 17 Drummond Place
1846 Dr Taylor, Pau, France
1840 Right Rev. Bishop Terrot, 9 Carlton Street
1843 Dr Allen Thomson, Prof. Anatomy, Univ., Glasgow
1866 *Dr Fraser Thomson, Perth
1842 James Thomson, Esq., C.E., Norfolk Square, Hyde Park, London
1863 *Dr Murray Thomson, Roorkee, East Indies
1864 *R. W. Thomson, Esq., C.E., 3 Moray Place 260
1847 *Sir William Thomson, Prof. Nat. Phil., Glasgow
1849 *William Thomas Thomson, Esq., Bonaly
1855 *Dr Wyville Thomson, Prof. Nat. Hist. and Geology, Belfast
1822 Sir W. C. Trevelyan, Bart., Wallington, Morpeth
1867 *William Turnbull, Esq., 14 Lansdowne Crescent
1861 *Professor Turner, 7 Brunswick Street, Hillside
1849 *Most Noble the Marquis of Tweeddale, K.T.
1867 *Peter Waddell, Esq., Claremont Park, Leith
1864 *Arthur Abney Walker, Esq., 32 Melville Street
1829 James Walker, Esq., W.S., Tunbridge Wells 270
1864 *William Wallace, Ph.D., Glasgow
1808 James Wardrop, Esq., London
1853 Dr James Watson, Bath
1866 *John K. Watson, Esq., 14 Blackford Road
1866 *Dr Patrick Heron Watson, 16 Charlotte Square
1862 *Rev. Robt. Boog Watson, Madeira, 4 Bruntsfield Place, Edinburgh
1840 Allan A. Maconochie Welwood, Esq. of Meadowbank and Pitliver
1868 *W. Williams, Esq., Veterinary College, Clyde Street
1858 *Dr Thomas Williamson, 40 Quality Street, Leith
1834 Dr Isaac Wilson 280
1847 Professor John Wilson, College
1863 *Dr J. G. Wilson, 9 Woodside Crescent, Glasgow
1864 *Dr Alexander Wood, 10 St Colme Street
1864 *Dr Andrew Wood, 9 Darnaway Street
1855 Dr Wright, Cheltenham
1864 *Robert S. Wyld, Esq., W.S., 19 Inverleith Row
1861 *James Young, Esq., Limefield, Mid-Calder
1863 *Dr John Young, Professor of Natural History, Glasgow 288

Fellows elected between the commencement of the Session and the 1st January of the following year are entered under the latter date, by which their Subscriptions are regulated:—Thus, Fellows elected in December 1867 have the date of 1868 prefixed to their names.

NON-RESIDENT MEMBER,

ELECTED UNDER THE OLD LAWS.

Sir Richard Griffiths, Bart., Dublin.

LIST OF HONORARY FELLOWS.

His Royal Highness the Prince of Wales.

FOREIGNERS (LIMITED TO THIRTY-SIX.)

	Louis Agassiz,	*Cambridge, Massachusetts.*
	J. B. A. L. Léonce Elie de Beaumont,	*Paris.*
	Robert Wilhelm Bunsen,	*Heidelberg.*
	Michel Eugene Chevreul,	*Paris.*
5	James D. Dana, LL.D.,	*Newhaven, Connecticut.*
	Jean Baptiste Dumas,	*Paris.*
	Charles Dupin,	*Do.*
	Christien Gottfried Ehrenberg,	*Berlin.*
	Pierre Marie Jean Flourens,	*Paris.*
10	Elias Fries,	*Upsala.*
	François Pierre Guillaume Guizot,	*Paris.*
	Wilhelm Karl Haidinger,	*Vienna.*
	Christopher Hansteen,	*Christiania.*
	Hermann Helmholtz,	*Heidelberg.*
15	Albert Kölliker,	*Wurzburg.*
	Johann Lamont,	*Munich.*
	Richard Lepsius,	*Berlin.*
	Rudolph Leuckart,	*Giessen.*
	Urbain Jean Joseph Leverrier,	*Paris.*
20	Baron Justus von Liebig,	*Munich.*
	Carl Friedrich Philip von Martius,	*Do.*
	Henry Milne-Edwards,	*Paris.*
	Theodore Mommsen,	*Berlin.*

	Prof. Benjamin Peirce,	*United States Survey.*
25	Adolphe Pictet,	*Geneva.*
	Lambert Adolphe Jacques Quetelet,	*Brussels.*
	M. Le Comte De Remusat,	*Paris.*
	Henri Victor Regnault,	*Do.*
	Augusta De la Rive,	*Geneva.*
30	Gustav Rose,	*Berlin.*
	Angelo Secchi,	*Rome.*
	Karl Theodore von Siebold,	*Munich.*
	Bernard Studer,	*Berne.*
34	Friedrich Wöhler,	*Gottingen.*

BRITISH SUBJECTS (LIMITED TO TWENTY BY LAW X.)

	John Couch Adams, Esq.,	*Cambridge.*
	George Biddell Airy, Esq.,	*Greenwich.*
	Thomas Carlyle, Esq.,	*London.*
	Arthur Cayley, Esq.,	*Cambridge.*
5	Charles Darwin, Esq.,	*Down, Bromley, Kent.*
	Thomas Graham, Esq.,	*London.*
	Sir John Frederick William Herschel, Bart.,	*Collingwood.*
	James Prescott Joule, LL D.,	*Cliffpoint, Higher Broughton, Manchester.*
	William Lassell, Esq.,	*Liverpool.*
10	Rev. Dr Humphrey Lloyd,	*Dublin.*
	Sir William E. Logan,	*London.*
	Sir Charles Lyell, Bart.,	*Do.*
	John Stuart Mill, Esq.,	*Do.*
	Sir Roderick Impey Murchison,	*Do.*
15	Richard Owen, Esq.,	*Do.*
	Lieut.-General Edward Sabine, R.A.,	*Do.*
	George Gabriel Stokes, Esq.,	*Cambridge.*
	William Henry Fox Talbot, Esq.,	*Lacock Abbey, Wiltshire.*
	Alfred Tennyson, Esq.,	*Freshwater, Isle of Wight.*
20	Sir Charles Wheatstone, D.C.L.,	*London.*

The following Public Institutions and Individuals are entitled to receive Copies of the Transactions and Proceedings of the Royal Society of Edinburgh :—

ENGLAND.

The British Museum.
The Bodleian Library, Oxford.
The University Library, Cambridge.

The Royal Society.
The Linnean Society.
The Society for the Encouragement of Arts.
The Geological Society.
The Royal Astronomical Society.
The Royal Asiatic Society.
The Zoological Society.
The Royal Society of Literature.
The Royal Horticultural Society.
The Royal Institution.
The Royal Geographical Society.
The Statistical Society.
The Institution of Civil Engineers.
The Institute of British Architects.
The Hydrographical Office, Admiralty.
The Medico-Chirurgical Society.
The Athenæum Club.
The Cambridge Philosophical Society.
The Manchester Literary and Philosophical Society.
The Yorkshire Philosophical Society.
The Chemical Society of London.
The Museum of Economic Geology.
The United Service Institution.
The Royal Observatory, Greenwich.
The Leeds Philosophical and Literary Society.
The Historic Society of Lancashire and Cheshire.
The Royal College of Surgeons of England.

SCOTLAND.

Edinburgh, University Library.
... Advocates' Library.
... College of Physicians.
Edinburgh, Highland and Agricultural Society.
... Royal Medical Society.
... Royal Physical Society.
... Royal Scottish Society of Arts.
Glasgow, University Library.
St Andrews, University Library.
Aberdeen, University Library.

IRELAND.

The Library of Trinity College, Dublin.
The Royal Irish Academy.

COLONIES, &c.

The Asiatic Society of Calcutta.
Library of Geological Survey, Calcutta.
The Literary and Historical Society of Toronto.
University of Sydney.

CONTINENT OF EUROPE.

Amsterdam, Royal Institute of Holland.
Berlin, Royal Academy of Sciences.
... Physical Society.
Berne, Society of Swiss Naturalists.
Bologna, Academy of Sciences.
Bonn, Cæsarean Academy of Naturalists.
Bordeaux Society of Physical and Natural Sciences.
Brussels, Royal Academy of Sciences.
Buda, Literary Society of Hungary.
Copenhagen, Royal Academy of Sciences.
Frankfort, the Senkenbergian Museum.
Geneva, Natural History Society.
Giessen, University Library.
Göttingen, University Library.
Haarlem, Natural History Society.
Jena, D. Gegerbaum, Editor of Zeitschrift Medicinisch-Physikalisch Gesellschaft.
Leipzig, Royal Saxon Academy.
Lille, Royal Society of Sciences.
Lisbon, Royal Academy of Sciences.

Lyons, Agricultural Society.
Milan, Royal Institute.
Moscow, Imperial Academy of Naturalists.
Munich, Royal Academy of Sciences of Bavaria (2 copies).
Neufchatel, Museum of Natural History.
Paris, Royal Academy of Sciences.
... Geographical Society.
... Royal Society of Agriculture.
.. Society for Encouragement of Industry.
... Geological Society of France.
... Ecole des Mines.
... Marine Depôt.
... Museum of Jardin des Plantes.
Rotterdam, Batavian Society of Experimental Philosophy.
St Petersburg, Imperial Academy of Sciences.
... Pulkowa Observatory.
Stockholm, Royal Academy of Sciences.
Turin, Royal Academy of Sciences.
Turin, M. Michelotti.
Upsala, Society of Sciences.
Venice, Royal Institute.
Vienna, Imperial Academy of Sciences.
... Geological Society.
... Geologico-Botanical Society.

UNITED STATES OF AMERICA.

Boston, the Bowditch Library.
... Academy of Arts and Sciences.
... Society of Natural History.
New York, State Library.
Philadelphia, American Philosophical Society.
... Academy of Natural Sciences.
Washington, the Smithsonian Institution.

(*All the Honorary and Ordinary Fellows of the Society are entitled to the Transactions and Proceedings.*)

The following Institutions and Individuals receive the Proceedings only:—

ENGLAND.

The Scarborough Philosophical Society.
The Whitby Philosophical Society.
The Newcastle Philosophical Society.
The Geological Society of Cornwall.
The Ashmolean Society of Oxford.
The Literary and Philosophical Society of Liverpool.

SCOTLAND.

The Philosophical Society of Glasgow.
The Botanical Society of Edinburgh.
The Geological Society of Edinburgh.
The Meteorological Society of Edinburgh.

IRELAND.

The Natural History Society of Dublin.

COLONIES.

The Literary and Philosophical Society of Quebec.
The Library of the Geological Survey, Canada.
The Literary Society of Madras.
China Branch of Asiatic Society, Hongkong.
North China Branch of the Royal Asiatic Society, Shanghae.
The Royal Society of Victoria.

CONTINENT OF EUROPE.

Utrecht, the Literary and Philosophical Society.
Paris, Editor of L'Institut.
Cherbourg, Society of Natural Sciences.
Sicily, Catania, Academia Govenia de Scienze Naturali.

UNITED STATES.

H. T. Parker, Esq., Harvard College, Cambridge.

TRANSACTIONS.

I.—*On Polyzomal Curves, otherwise the Curves* $\sqrt{U} + \sqrt{V} + \&c. = 0$.

By Professor CAYLEY. Communicated by Professor TAIT.

(Read 16th December 1867.)

IF U, V, &c., are rational and integral functions $(*)(x, y, z)^r$, all of the same degree r, in regard to the co-ordinates (x, y, z), then $\sqrt{U} + \sqrt{V} + \&c.$ is a polyzome, and the curve $\sqrt{U} + \sqrt{V} + \&c. = 0$ a polyzomal curve. Each of the curves $\sqrt{U} = 0$, $\sqrt{V} = 0$, &c. (or say the curves $U = 0$, $V = 0$, &c.) is, on account of its relation of circumscription to the curve $\sqrt{U} + \sqrt{V} + \&c. = 0$, considered as a girdle thereto (ζῶμα), and we have thence the term "zome" and the derived expressions "polyzome," "zomal," &c. If the number of the zomes $\sqrt{U}, \sqrt{V}$, &c. be $= \nu$, then we have a ν-zome, and corresponding thereto a ν-zomal curve; the curves $U = 0$, $V = 0$, &c., are the zomal curves or zomals thereof. The cases $\nu = 1$, $\nu = 2$, are not, for their own sake, worthy of consideration; it is in general assumed that ν is $= 3$ at least. It is sometimes convenient to write the general equation in the form $\sqrt{lU} + \&c. = 0$, where l, &c. are constants. The Memoir contains researches in regard to the general ν-zomal curve; the branches thereof, the order of the curve, its singularities, class, &c.; also in regard to the ν-zomal curve $\sqrt{l(\Theta + L\,\Phi)} + \&c. = 0$, where the zomal curves $\Theta + L\,\Phi = 0$, all pass through the points of intersection of the same two curves $\Theta = 0$, $\Phi = 0$ of the orders r and $r-s$ respectively; included herein we have the theory of the depression of order as arising from the ideal factor or factors of a branch or branches. A general theorem is given of "the decomposition of a tetrazomal curve," viz., if the equation of the curve be $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} + \sqrt{pT} = 0$; then if U, V, W, T are in involution, that is, connected by an identical equation $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W + \mathrm{d}T = 0$, and if l, m, n, p, satisfy the condition $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0$, the tetrazomal curve breaks up into two trizomal curves, each expressible by means of any three of the four functions U, V, W, T; for example, in the form $\sqrt{l'U} + \sqrt{m'V} + \sqrt{p'T} = 0$. If, in this theorem, we take $p = 0$, then the original curve is the trizomal $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$, T is any function $= -\frac{1}{\mathrm{d}}(\mathrm{a}U + \mathrm{b}V + \mathrm{c}W)$, where, considering l, m, n as given, a, b, c are quantities subject only to the condition $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, and we have the theorem of "the variable zomal of a trizomal curve," viz., the equation of the trizomal $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$, may be expressed by means of any two of the three functions U, V, W, and of a function T determined as above, for example in the form $\sqrt{l'U} + \sqrt{m'V} + \sqrt{n'T} = 0$; whence also it may be expressed in terms of three new functions T, determined as above. This theorem, which occupies a prominent position in the whole theory, was suggested to me by Mr Casey's theorem, presently referred to, for the construction of a bicircular quartic as the envelope of a variable circle.

In the ν-zomal curve $\sqrt{l(\Theta + L\Phi)} + \&\text{c.} = 0$, if $\Theta = 0$ be a conic, $\Phi = 0$ a line, the zomals $\Theta + L\Phi = 0$, &c. are conics passing through the same two points $\Theta = 0$, $\Phi = 0$, and there is no real loss of generality in taking these to be the circular points at infinity—that is, in taking the conics to be circles. Doing this, and using a special notation $\mathsf{A}^{\circ} = 0$ for the equation of a circle having its centre at a given point A, and similarly $\mathsf{A} = 0$ for the equation of an evanescent circle, or say of the point A, we have the ν-zomal curve $\sqrt{l\mathsf{A}^{\circ}} + \&\text{c.} = 0$, and the more special form $\sqrt{l\mathsf{A}} + \&\text{c.} = 0$. As regards the last-mentioned curve, $\sqrt{l\mathsf{A}} + \&\text{c.} = 0$, the point A to which the equation $\mathsf{A} = 0$ belongs, is a focus of the curve, viz., in the case $\nu = 3$, it is an ordinary focus, and in the case $\nu > 3$, it is a special kind of focus, which, if the term were required, might be called a foco-focus; the Memoir contains an explanation of the general theory of the foci of plane curves. For $\nu = 3$, the equation $\sqrt{l\mathsf{A}} + \sqrt{m\mathsf{B}} + \sqrt{n\mathsf{C}} = 0$ is really equivalent to the apparently more general form $\sqrt{l\mathsf{A}^{\circ}} + \sqrt{m\mathsf{B}^{\circ}} + \sqrt{n\mathsf{C}^{\circ}} = 0$. In fact, this last is in general a bicircular quartic, and, in regard to it, the before-mentioned theorem of the variable zomal becomes Mr Casey's theorem, that "the bicircular quartic (and, as a particular case thereof, the circular cubic) is the envelope of a variable circle having its centre on a given conic and cutting at right angles a given circle." This theorem is a sufficient basis for the complete theory of the trizomal curve $\sqrt{l\mathsf{A}^{\circ}} + \sqrt{m\mathsf{B}^{\circ}} + \sqrt{n\mathsf{C}^{\circ}} = 0$; and it is thereby very easily seen that the curve $\sqrt{l\mathsf{A}^{\circ}} + \sqrt{m\mathsf{B}^{\circ}} + \sqrt{n\mathsf{C}^{\circ}} = 0$ can be represented by an equation $\sqrt{l'\mathsf{A}'} + \sqrt{m'\mathsf{B}'} + \sqrt{n'\mathsf{C}'} = 0$. But for $\nu > 3$ this is not so, and the curve $\sqrt{l\mathsf{A}} + \&\text{c.} = 0$ is only a particular form of the curve $\sqrt{l\mathsf{A}^{\circ}} + \&\text{c.} = 0$; and the discussion of this general form is scarcely more difficult than that of the special form $\sqrt{l\mathsf{A}} + \&\text{c.} = 0$, included therein. The investigations in relation to the theory of foci, and in particular to that of the foci of the circular cubic and bicircular quartic, precede in the Memoir the theories of the trizomal curve

$\sqrt{l\mathsf{A}^\circ}+\sqrt{m\mathsf{B}^\circ}+\sqrt{n\mathsf{C}^\circ}=0$, and the tetrazomal curve $\sqrt{l\mathsf{A}^\circ}+\sqrt{m\mathsf{B}^\circ}+\sqrt{n\mathsf{C}^\circ}+\sqrt{p\mathsf{D}^\circ}=0$, to which the concluding portions relate. I have accordingly divided the Memoir into four parts, viz., these are—Part I., On Polyzomal Curves in general; Part II., Subsidiary Investigations; Part III., On the Theory of Foci; and Part IV., On the Trizomal and Tetrazomal Curves where the zomals are circles. There is, however, some necessary intermixture of the theories treated of, and the arrangement will appear more in detail from the headings of the several articles. The paragraphs are numbered continuously through the Memoir. There are four Annexes, relating to questions which it seemed to me more convenient to treat of thus separately.

It is right that I should explain the very great extent to which, in the composition of the present Memoir, I am indebted to Mr Casey's researches. His Paper "On the Equations and Properties (1.) of the System of Circles touching three circles in a plane; (2.) of the System of Spheres touching four spheres in space; (3.) of the System of Circles touching three circles on a sphere; (4.) on the System of Conics inscribed in a conic and touching three inscribed conics in a plane," was read to the Royal Irish Academy, April 9, 1866, and is published in their "Proceedings." The fundamental theorem for the equation of the pairs of circles touching three given circles was, previous to the publication of the paper, mentioned to me by Dr Salmon, and I communicated it to Professor Cremona, suggesting to him the problem solved in his letter of March 3, 1866, as mentioned in my paper, "Investigations in connection with Casey's Equation," "Quarterly Math. Journal," t. viii. 1867, pp. 334–341, and as also appears, Annex No. IV. of the present Memoir.

In connection with this theorem, I communicated to Mr Casey, in March or April 1867, the theorem No. 164 of the present Memoir, that for any three given circles, centres A, B, C, the equation $\overline{BC}\sqrt{\mathsf{A}^\circ}+\overline{CA}\sqrt{\mathsf{B}^\circ}+\overline{AB}\sqrt{\mathsf{C}^\circ}=0$ (where $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, denote the mutual distances of the points A, B, C) belongs to a Cartesian. Mr Casey, in a letter to me dated 30th April 1867, informed me of his own mode of viewing the question as follows:—"The general equation of the second order $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2=0$, where α, β, γ are circles, is a bicircular quartic. If we take the equation $(a, b, c, f, g, h)(\lambda, \mu, \nu)^2=0$ in tangential co-ordinates (that is, when λ, μ, ν are perpendiculars let fall from the centres of α, β, γ on any line), it denotes a conic; denoting this conic by F, and the circle which cuts α, β, γ orthogonally by J, I proved that, if a variable circle moves with its centre on F, and if it cuts J orthogonally, its envelope will be the bicircular quartic whose equation is that written down above;" and among other consequences, he mentions that the foci of F are the double foci of the quartic, and the points in which J cuts F single foci of the quartic, and also the theorem which I had sent him as to the Cartesian, and he refers to his Memoir on Bicircular Quartics as then nearly finished. An Abstract of the Memoir as read before the Royal Irish

Academy, 10th February 1867, and published in their "Proceedings," pp. 44, 45, contains the theorems mentioned in the letter of 30th April, and some other theorems. It is not necessary that I should particularly explain in what manner the present Memoir has been, in the course of writing it, added to or altered in consequence of the information which I have thus had of Mr Casey's researches: it is enough to say that I have freely availed myself of such information, and that there is no question as to Mr Casey's priority in anything which there may be in common in his memoir on Bicircular Quartics and in the present Memoir.

PART I. (Nos. 1 to 55).—ON POLYZOMAL CURVES IN GENERAL.

Definition and Preliminary Remarks—Art. Nos. 1 to 4.

1. As already mentioned, U, V, &c. denote rational and integral functions $(*)(x, y, z)^r$, all of the same degree r in the co-ordinates (x, y, z), and the equation

$$\sqrt{U} + \sqrt{V} + \text{\&c.} = 0$$

then belongs to a polyzomal curve, viz., if the number of the zomes $\sqrt{U}$, $\sqrt{V}$, &c., is $= \nu$, then we have a ν-zomal curve. The radicals, or any of them, may contain rational factors, or be of the form $P\sqrt{Q}$; but in speaking of the curve as a ν-zomal, it is assumed that any two terms, such as $P\sqrt{Q} + P'\sqrt{Q}$, involving the same radical $\sqrt{Q}$, are united into a single term, so that the number of distinct radicals is always $= \nu$; in particular (r being even), it is assumed that there is only one rational term P. But the ordinary case, and that which is almost exclusively attended to, is that in which the radicals $\sqrt{U}$, $\sqrt{V}$, &c. are distinct irreducible radicals without rational factors.

2. The curves $U = V = 0$, &c. are said to be the zomal curves, or simply the zomals of the polyzomal curve $\sqrt{U} + \sqrt{V} + \text{\&c.} = 0$; more strictly, the term zomal would be applied to the functions U, V, &c. It is to be noticed, that although the form $\sqrt{U} + \sqrt{V} + \text{\&c.} = 0$ is equally general with the form $\sqrt{lU} + \sqrt{mV} + \text{\&c.} = 0$ (in fact, in the former case, the functions U, V, &c., are considered as implicitly containing the constant factors l, m, &c., which are expressed in the latter case), yet it is frequently convenient to express these factors, and thus write the equation in the form $\sqrt{lU} + \sqrt{mV} + \text{\&c.}$ For instance, in speaking of any given curves $U = 0$, $V = 0$, &c., we are apt, disregarding the constant factors which they may involve, to consider U, V, &c. as given functions; but in this case the general equation of the polyzomal with the zomals $U = 0$, $V = 0$, &c., is of course $\sqrt{lU} + \sqrt{mV} + \text{\&c.} = 0$.

3. Anticipating in regard to the cases $\nu = 1$, $\nu = 2$, the remark which will be presently made in regard to the ν-zomal, that $\sqrt{U} + \sqrt{V} + \text{\&c.} = 0$ is the curve represented by the rationalised form of this equation, the monozomal curve $\sqrt{U} = 0$ is merely the curve $U = 0$, viz., this is any curve whatever $U = 0$ of the

order r; and similarly, the bizomal curve $\sqrt{U}+\sqrt{V}=0$ is merely the curve $U-V=0$, viz. this is any curve whatever $\Omega=0$, of the order r; the zomal curves $U=0$, $V=0$, taken separately, are not curves standing in any special relation to the curve in question $\Omega=0$, but $U=0$ may be any curve whatever of the order r, and then $V=0$ is a curve of the same order r, in involution with the two curves $\Omega=0$, $U=0$; we may, in fact, write the equation $\Omega=0$ under the bizomal form $\sqrt{U}+\sqrt{\Omega+U}=0$. In the case r even, we may, however, notice the bizomal curve $P+\sqrt{U}=0$ (P a rational function of the degree $\tfrac{1}{2}r$); the rational equation is here $\Omega=U-P^2=0$, that is $U=\Omega+P^2$, viz., P is any curve whatever of the order $\tfrac{1}{2}r$, and $U=0$ is a curve of the order r, touching the given curve $\Omega=0$ at each of its $\tfrac{1}{2}r^2$ intersections with the curve $P=0$. I further remark that the order of the ν-zomal curve $\sqrt{V}+\text{\&c.}=0$ is $=2^{\nu-2}r$; this is right in the case of the bizomal curve $\sqrt{U}+\sqrt{V}=0$, the order being $=r$, but it fails for the monozomal curve $\sqrt{U}=0$, the order being in this case r, instead of $\tfrac{1}{2}r$, as given by the formula. The two unimportant and somewhat exceptional cases $\nu=1$, $\nu=2$, are thus disposed of, and in all that follows (except in so far as this is in fact applicable to the cases just referred to), ν may be taken to be $=3$ at least.

4. It is to be throughout understood that by the curve $\sqrt{U}+\sqrt{V}+\text{\&c.}=0$ is meant the curve represented by the rationalised equation—

$$\text{Norm }(\sqrt{U}+\sqrt{V}+\text{\&c.})=0$$

viz. the Norm is obtained by attributing to all but one of the zomes $\sqrt{U}$, $\sqrt{V}$, &c., each of the two signs $+$, $-$, and multiplying together the several resulting values of the polyzome; in the case of a ν-zomal curve, the number of factors is thus $=2^{\nu-1}\,r$ (whence, as each factor is of the degree $\tfrac{1}{2}r$, the order of the curve is $2^{\nu-1}\,.\,\tfrac{1}{2}r$, $=2^{\nu-2}\,r$, as mentioned above). I expressly mention that, as regards the polyzomal curve, we are not in any wise concerned with the signs of the radicals, which signs are and remain essentially indeterminate; the equation $\sqrt{U}+\sqrt{V}+\text{\&c.}=0$, is a mere symbol for the rationalised equation, Norm $(\sqrt{U}+\sqrt{V}+\text{\&c.})=0$.

The Branches of a Polyzomal Curve—Art. Nos. 5 to 12.

5. But we may in a different point of view attend to the signs of the radicals; if for all values of the co-ordinates we take the symbol $\sqrt{\overline{\overline{\ \ }}}$, and consider $\sqrt{\overline{\overline{U}}}$, $\sqrt{\overline{\overline{V}}}$, &c. as signifying determinately, say the *positive* values of $\sqrt{U}$, $\sqrt{V}$, &c.; then each of the several equations $\pm\sqrt{\overline{\overline{U}}}\pm\sqrt{\overline{\overline{V}}}+\text{\&c.}=0$, or, fixing at pleasure one of the signs, suppose that prefixed to $\sqrt{\overline{\overline{U}}}$, then each of the several equations $\sqrt{\overline{\overline{U}}}\pm\sqrt{\overline{\overline{V}}}\pm\text{\&c.}=0$, will belong to a *branch* of the polyzomal curve: a ν-zomal curve has thus $2^{\nu-1}$ branches corresponding to the $2^{\nu-1}$ values respec-

tively of the polyzome. The separation of the branches depends on the precise fixation of the significations of $\sqrt{U}$, $\sqrt{V}$, &c., and in regard hereto some further explanation is necessary.

6. When U is real and positive, $\sqrt{U}$ may be taken to be, in the ordinary sense, the positive value of $\sqrt{U}$, and so when U is real and negative, $\sqrt{U}$ may be taken to be $= i$ into the positive value of $\sqrt{-U}$; and the like as regards $\sqrt{V}$, &c. The functions U, V, &c., are assumed to be real functions of the co-ordinates; hence, for any real values of the co-ordinates, U, V, &c. are real positive or negative quantities, and the significations of $\sqrt{U}$, $\sqrt{V}$, &c. are completely determined.

7. But the co-ordinates may be imaginary. In this case the functions U, V, &c. will for any given values of the co-ordinates acquire each of them a determinate, in general imaginary, value. If for all real values whatever of α, β, we select once for all one of the two opposite values of $\sqrt{\alpha + \beta i}$, *calling* it the *positive* value, and representing it by $\sqrt{\alpha + \beta i}$, then, for any particular values of the co-ordinates, U being $= \alpha + \beta i$, the value of $\sqrt{U}$ may be taken to be $= \sqrt{\alpha + \beta i}$; and the like as regards $\sqrt{V}$, &c. $\sqrt{U}$, $\sqrt{V}$, &c. have thus each of them a determinate signification for any values whatever, real or imaginary, of the co-ordinates. The co-ordinates of a given point on the curve $\sqrt{U} + \sqrt{V} + \text{\&c.} = 0$, will in general satisfy only one of the equations $\sqrt{U} \pm \sqrt{V} \pm \text{\&c.} = 0$; that is, the point will belong to one (but in general only one) of the $2^{\nu-1}$ branches of the curve; the entire series of points the co-ordinates of which satisfy any one of the $2^{\nu-1}$ equations, will constitute the branch corresponding to that equation.

8. The signification to be attached to the expression $\sqrt{\alpha + \beta i}$ should agree with that previously attached to the like symbol in the case of a positive or negative real quantity; and it should, as far as possible, be subject to the condition of continuity, viz., as $\alpha + \beta i$ passes continuously to $\alpha' + \beta' i$, so $\sqrt{\alpha + \beta i}$ should pass continuously to $\sqrt{\alpha' + \beta' i}$; but (as is known) it is not possible to satisfy universally this condition of continuity; viz., if for facility of explanation we consider (α, β) as the co-ordinates of a point in a plane, and imagine this point to describe a closed curve surrounding the origin or point $(0, 0)$, then it is not possible so to define $\sqrt{\alpha + \beta i}$ that this quantity, varying continuously as the point moves along the curve, shall, when the point has made a complete circuit, resume its original value. The signification to be attached to $\sqrt{\alpha + \beta i}$ is thus in some measure arbitrary, and it would appear that the division of the curve into branches is affected by a corresponding arbitrariness, but this arbitrariness relates only to the imaginary branches of the curve: the notion of a real branch is perfectly definite.

9. It would seem that a branch may be impossible for any series whatever of points real or imaginary. Thus, in the bizomal curve $\sqrt{U} + \sqrt{V} = 0$, the branch $\sqrt{U} + \sqrt{V} = 0$ is impossible. In fact, for any point whatever, real or

imaginary, of the curve, we have $U = V$, and therefore $\sqrt{U} = \sqrt{V}$; the point thus belongs to the other branch $\sqrt{U} - \sqrt{V} = 0$, not to the branch $\sqrt{U} + \sqrt{V} = 0$; the only points belonging to the last-mentioned branch are the isolated points for which simultaneously $\sqrt{U} = 0$, $\sqrt{V} = 0$; viz., the points of intersection of the two curves $U = 0$, $V = 0$.

10. It is not clear to me whether the case is the same in regard to the branch $\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$ of a trizomal curve. In fact, for each point of the curve $\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$ we have $(U - V - W)^2 = 4\,VW$, and therefore, $U - V - W = \pm 2\sqrt{V}\sqrt{W}$; there may very well be points for which the sign is +; that is, points for which $U = V + W + 2\sqrt{V}\sqrt{W}$, and for these points we have $\pm\sqrt{U} = \sqrt{V} + \sqrt{W}$; for real values of the co-ordinates the sign on the left hand must be + (for otherwise the two sides would have opposite signs), but there is no apparent reason, or at least no obviously apparent reason, why this should be so for imaginary values of the co-ordinates, and if the sign be in fact —, then the point will belong to the branch $\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$.

11. But the branch in question is clearly impossible for any series of real points; so that, leaving it an open question whether the epithet "impossible" is to be understood to mean impossible for any series of real points (that is, as a mere synonym of imaginary), or whether it is to mean impossible for any series of points, real or imaginary, whatever, I say that in a ν-zomal curve some of the branches are or may be impossible, and that there is at least one impossible branch, viz. the branch $\sqrt{U} + \sqrt{V} + \&\text{c.} = 0$.

12. For the purpose of referring to any branch of a polyzomal curve it will be convenient to consider $\sqrt{U}$ as signifying determinately $+\sqrt{U}$, or else $-\sqrt{U}$; and the like as regards $\sqrt{V}$, &c., but without any identity or relation between the signs prefixed to the $\sqrt{U}$, $\sqrt{V}$, &c., respectively; the equation $\sqrt{U} + \sqrt{V} + \&\text{c.} = 0$, so understood, will denote determinately some one (that is, any one at pleasure) of the equations $\sqrt{U} \pm \sqrt{V} \pm \&\text{c.} = 0$, and it will thus be the equation of some one (that is, any one at pleasure) of the branches of the polyzomal curve — all risk of ambiguity which might otherwise exist will be removed if we speak either of the *curve* $\sqrt{U} + \sqrt{V}$, &c. $= 0$, or else of the *branch* $\sqrt{U} + \sqrt{V} + \&\text{c.} = 0$. Observe that by the foregoing convention, when only one branch is considered, we avoid the necessity of any employment of the sign ±, or of the sign —; but when two or more branches are considered in connection with each other, it is necessary to employ the sign — with one or more of the radicals $\sqrt{U}$, $\sqrt{V}$, &c.; thus in the trizomal curve $\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$, we may have to consider the branches $\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$, $\sqrt{U} + \sqrt{V} - \sqrt{W} = 0$; viz., either of these equations apart from the other denotes any one branch at pleasure of the curve, but when the branch represented by the one equation is fixed, then the branch represented by the other equation is also fixed.

The Points common to Two Branches of a Polyzomal Curve—Art. Nos. 13 to 17.

13. I consider the points which are situate simultaneously on two branches of the ν-zomal curve $\sqrt{U} + \sqrt{V} + \text{\&c.} = 0$. The equations of the two branches may be taken to be

$$\sqrt{U} + \text{\&c.} + (\sqrt{W} + \text{\&c.}) = 0,$$
$$\sqrt{U} + \text{\&c.} - (\sqrt{W} + \text{\&c.}) = 0,$$

viz., fixing the significations of $\sqrt{U}$, $\sqrt{V}$, $\sqrt{W}$, &c. in such wise that in the equation of one branch these shall each of them have the sign $+$, we may take $\sqrt{U}$, &c. to be those radicals which, in the equation of the other branch, have the sign $+$, and $\sqrt{W}$, &c. to be those radicals which have the sign $-$. The foregoing equations break up into the more simple equations

$$\sqrt{U} + \text{\&c.} = 0,\ \sqrt{W} + \text{\&c.} = 0,$$

which are the equations of certain *branches* of the *curves* $\sqrt{U} + \text{\&c.} = 0$, and $\sqrt{W} + \text{\&c.} = 0$, respectively, and conversely each of the intersections of these two curves is a point situate simultaneously on some two branches of the original ν-zomal curve $\sqrt{U} + \sqrt{V} + \text{\&c.} = 0$. Hence, partitioning in any manner the ν-zome $\sqrt{U} + \sqrt{V} + \text{\&c.}$ into an α-zome, $\sqrt{U} + \text{\&c.}$ and a β-zome $\sqrt{W} + \text{\&c.}$ $(\alpha + \beta = \nu)$, and writing down the equations

$$\sqrt{U} + \text{\&c.} = 0,\ \sqrt{W} + \text{\&c.} = 0$$

of an α-zomal curve and a β-zomal curve respectively, each of the intersections of these two curves is a point situate simultaneously on two branches of the ν-zomal curve; and the points situate simultaneously on two branches of the ν-zomal curve are the points of intersection of the several pairs of an α-zomal curve and a β-zomal curve, which can be formed by any bipartition of the ν-zome.

14. There are two cases to be considered:—First, when the parts are $1, \nu - 1$ ($\nu - 1$ is > 1, except in the case $\nu = 2$, which may be excluded from consideration), or say when the ν-zome is partitioned into a *zome* and *antizome*. Secondly, when the parts α, β, are each > 1 (this implies $\nu = 4$ at least), or say when the ν-zome is partitioned into a pair of *complementary parazomes*.

15. To fix the ideas, take the tetrazomal curve $\sqrt{U} + \sqrt{V} + \sqrt{W} + \sqrt{T} = 0$, and consider first a point for which $\sqrt{U} = 0$, $\sqrt{V} + \sqrt{W} + \sqrt{T} = 0$. The Norm is the product of $(2^3 =)$ 8 factors; selecting hereout the factors

$$\sqrt{U} + \sqrt{V} + \sqrt{W} + \sqrt{T},$$
$$\sqrt{U} - \sqrt{V} - \sqrt{W} - \sqrt{T},$$

let the product of these

$$= U - (\sqrt{V} + \sqrt{W} + \sqrt{T})^2$$

be called F, and the product of the remaining six factors be called G; the rationalised equation of the curve is therefore $FG = 0$. The derived equation is $GdF + FdG = 0$; at the point in question $\sqrt{U} = 0$, $\sqrt{V} + \sqrt{W} + \sqrt{T} = 0$; G and dG are each of them finite (that is, they neither vanish nor become infinite), but we have

$$F = 0,\ dF = dU - (\sqrt{V} + \sqrt{W} + \sqrt{T})\,(dV \div \sqrt{V} + dW \div \sqrt{W} + dT \div \sqrt{T}),\ = dU,$$

and the derived equation is thus $GdU = 0$, or simply $dU = 0$. It thus appears that the point in question is an ordinary point on the tetrazomal curve; and, further, that the tetrazomal curve is at this point touched by the zomal curve $U = 0$. And similarly, each of the points of intersection of the two curves $\sqrt{U} = 0$, $\sqrt{V} + \sqrt{W} + \sqrt{T} = 0$, is an ordinary point on the tetrazomal curve; and the tetrazomal curve is at each of these points touched by the zomal curve $U = 0$.

16. Consider, secondly, a point for which $\sqrt{U} + \sqrt{V} = 0$, $\sqrt{W} + \sqrt{T} = 0$; to form the Norm, taking in this case the two factors

$$\sqrt{U} + \sqrt{V} + \sqrt{W} + \sqrt{T},$$
$$\sqrt{U} + \sqrt{V} - \sqrt{W} - \sqrt{T},$$

let their product

$$= (\sqrt{U} + \sqrt{V})^2 - (\sqrt{W} + \sqrt{T})^2$$

be called F, and the product of the remaining six factors be called G; the rationalised equation is $FG = 0$, and the derived equation is $FdG + GdF = 0$. At the point in question G and dG are each of them finite (that is, they neither vanish nor become infinite), but we have

$$F=0,\ dF=(\sqrt{U}+\sqrt{V})\,(dU\div\sqrt{U}+dV\div\sqrt{V})-(\sqrt{W}+\sqrt{T})\,(dW\div\sqrt{W}+dT\div\sqrt{T}),\ =0,$$

that is, the derived equation becomes identically $0 = 0$; the point in question is thus a singular point, and it is easy to see that it is in fact a node, or ordinary double point, on the tetrazomal curve. And similarly, each of the points of intersection of the two curves $\sqrt{U} + \sqrt{V} = 0$, $\sqrt{W} + \sqrt{T} = 0$ is a node on the tetrazomal curve.

17. The proofs in the foregoing two examples respectively are quite general, and we may, in regard to a ν-zomal curve, enunciate the results as follows, viz., in a ν-zomal curve, the points situate simultaneously on two branches are either the intersections of a zomal curve and its antizomal curve, or else they are the intersections of a pair of complementary parazomal curves. In the former case, the points in question are ordinary points on the ν-zomal, but they are points of contact of the ν-zomal with the zomal; it may be added, that the intersections of the zomal and antizomal, each reckoned twice, are *all* the intersections of the ν-zomal and zomal. In the latter case, the points in question are nodes of the ν-zomal; it may be added, that the ν-zomal has not, *in general*, any nodes other than the points which are thus the intersections of a pair of complementary parazomals, and that it has not *in general* any cusps.

Singularities of a ν-zomal Curve—Art. Nos. 18 to 21.

18. It has been already shown that the order of the ν-zomal curve is $= 2^{\nu-2}r$. Considering the case where ν is $= 3$ at least, the curve, as we have just seen, has contacts with each of the zomal curves, and it has also nodes. I proceed to determine the number of these contacts and nodes respectively.

19. Consider first the zomal curve $U = 0$, and its antizomal $\sqrt{V} + \sqrt{W} + \&c. = 0$, these are curves of the orders r and $2^{\nu-3}r$ respectively, and they intersect therefore in $2^{\nu-3}r^2$ points. Hence the ν-zomal touches the zomal in $2^{\nu-3}r^2$ points, and reckoning each of these twice, the number of intersections is $= 2^{\nu-2}r^2$, viz., these are all the intersections of the ν-zomal with the zomal $U = 0$. The number of contacts of the ν-zomal with the several zomals $U = 0$, $V = 0$, &c., is of course $= 2^{\nu-3}r^2\nu$.

20. Considering next a pair of complementary parazomal curves, an α-zomal and a β-zomal respectively ($\alpha + \beta = \nu$), these are of the orders $2^{\alpha-2}r$ and $2^{\beta-2}r$ respectively, and they intersect therefore in $2^{\alpha+\beta-4}r^2 = 2^{\nu-4}r^2$ points, nodes of the ν-zomal. This number is independent of the particular partition (α, β), and the ν-zomal has thus this same number, $2^{\nu-4}r^2$, of nodes in respect of each pair of complementary parazomals; hence the total number of nodes is $= 2^{\nu-4}r^2$ into the number of pairs of complementary parazomals. For the partition (α, β) the number of pairs is $= [\nu]^\nu \div [\alpha]^\alpha[\beta]^\beta$, or when $\alpha = \beta$, which of course implies ν even, it is one-half of this; extending the summation from $\alpha = 2$ to $\alpha = \nu - 2$, each pair is obtained twice, and the number of pairs is thus $= \tfrac{1}{2}\Sigma\left\{[\nu]^\nu \div [\alpha]^\alpha[\beta]^\beta\right\}$; the sum extended from $\alpha = 0$ to $\alpha = \nu$ is $(1 + 1)^\nu, = 2^\nu$, but we thus include the terms $1, \nu, \nu, 1$, which are together $= 2\nu + 2$, hence the correct value of the sum is $= 2^\nu - 2\nu - 2$, and the number of pairs is the half of this $= 2^{\nu-1} - \nu - 1$. Hence the number of nodes of the ν-zomal curve is $= (2^{\nu-1} - \nu - 1)\,2^{\nu-4}r^2$.

21. The ν-zomal is thus a curve of the order $2^{\nu-2}r$, with $(2^{\nu-1} - \nu - 1)\,2^{\nu-4}r^2$ nodes, but without cusps; the class is therefore

$$= 2^{\nu-3}r\left[(\nu + 1)\,r - 2\right],$$

and the deficiency is

$$= 2^{\nu-4}r\left[(\nu + 1)\,r - 6\right] + 1\,.$$

These are the general expressions, but even when the zomal curves $U = 0$, $V = 0$, &c., are given, then writing the equation of the ν-zomal under the form $\sqrt{lU} + \sqrt{mV} + \&c. = 0$, the constants $l : m :$ &c., may be so determined as to give rise to nodes or cusps which do not occur in the general case; the formulæ will also undergo modification in the particular cases next referred to.

Special Case where all the Zomals have a Common Point or Points—Art. Nos. 22 to 27.

22. Consider the case where the zomals $U = 0$, $V = 0$ have all of them any number, say k, of common intersections—these may be referred to simply as the common points. Each common point is a $2^{\nu-2}$-tuple point on the ν-zomal curve: it is on each zomal an ordinary point, and on each antizomal a $2^{\nu-3}$-tuple point, and on any α-zomal parazomal a $2^{\alpha-2}$-tuple point. Hence, considering first the intersections of any zomal with its antizomal, the common point reckons as $2^{\nu-3}$ intersections, and the k common points reckon as $2^{\nu-3}\,k$ intersections; the number of the remaining intersections is therefore $= 2^{\nu-3}(r^2 - k)$, and the zomal touches the ν-zomal in each of these points. The intersections of the zomal with the ν-zomal are the k common points, each of them a $2^{\nu-2}$-tuple point on the ν-zomal, and therefore reckoning together as $2^{\nu-2}k$ intersections; and the $2^{\nu-3}(r^2 - k)$ points of contact, each reckoning twice, and therefore together as $2^{\nu-2}(r^2 - k)$ intersections $(2^{\nu-2}k + 2^{\nu-2}(r^2 - k) = 2^{\nu-2}r^2, = r\,.\,2^{\nu-2}r)$; the total number of contacts with the zomals $U = 0$, $V = 0$, &c., is thus $= 2^{\nu-3}(r^2 - k)\,\nu$.

23. Secondly, considering any pair of complementary parazomals, an α-zomal and a β-zomal, each of the common points, being a $2^{\alpha-2}$-tuple point and a $2^{\beta-2}$-tuple point on the two curves respectively, counts as $2^{\alpha+\beta-4}, = 2^{\nu-4}$ intersections, and the k common points count as $2^{\nu-4}k$ intersections; the number of the remaining intersections is therefore $= 2^{\nu-4}(r^2 - k)$, each of which is a node on the ν-zomal curve; and we have thus in all $2^{\nu-4}(2^{\nu-1} - \nu - 1)(r^2 - k)$ nodes.

24. There are, besides, the k common points, each of them a $2^{\nu-2}$-tuple point on the ν-zomal, and therefore each reckoning as $\frac{1}{2}2^{\nu-2}(2^{\nu-2} - 1), = 2^{2\nu-5} - 2^{\nu-3}$ double points, or together as $(2^{2\nu-5} - 2^{\nu-3})k$ double points. Reserving the term node for the above-mentioned nodes or proper double points, and considering, therefore, the double points (dps.) as made up of the nodes and of the $2^{\nu-2}$-tuple points, the total number of dps. is thus

$$2^{\nu-4}(2^{\nu-1} - \nu - 1)(r^2 - k) + (2^{2\nu-5} - 2^{\nu-3})k,$$

$$= 2^{\nu-4}(2^{\nu-1} - \nu - 1)\,r^2 + (\nu + 1\,2^{\nu-4} - 2^{\nu-3})k\,;$$

or finally this is

$$= 2^{\nu-4}\big\{(2^{\nu-1} - \nu - 1)\,r^2 + (\nu - 1)\big\};$$

so that there is a gain $= 2^{\nu-4}(\nu - 1)k$ in the number of dps. arising from the k common points. There is, of course, in the class a diminution equal to twice this number, or $2^{\nu-3}(\nu - 1)k$; and in the deficiency a diminution equal to this number, or $2^{\nu-4}(\nu - 1)k$.

25. The zomal curves $U = 0$, $V = 0$, &c., may all of them pass through the same ν^2 points; we have then $k = r^2$, and the expression for the number of dps

is $= (2^{2\nu-5} - 2^{\nu-3})r^2$, viz., this is $= \frac{1}{2}\, 2^{\nu-2}\,(2^{\nu-2} - 1)r^2$. But in this case the dps. are nothing else than the r^2 common points, each of them a $2^{\nu-2}$-tuple point, the ν-zomal curve in fact breaking up into a system of $2^{\nu-2}$ curves of the order r, each passing through the r^2 common points. This is easily verified, for if $\Theta = 0$, $\Phi = 0$ are some two curves of the order r, then, in the present case, the zomal curves are curves in involution with these curves; that is, they are curves of the form $l\Theta + l'\Phi = 0$, $m\Theta + m'\Phi = 0$, &c., and the equation of the ν-zomal curve is

$$\sqrt{l\Theta + l'\Phi} + \sqrt{m\Theta + m'\Phi} + \&\text{c.} = 0 .$$

The rationalised equation is obviously an equation of the degree $2^{\nu-2}$ in Θ, Φ, giving therefore a constant value for the ratio $\Theta : \Phi$; calling this q, or writing $\Theta = q\Phi$, we have

$$\sqrt{lq + l'} + \sqrt{mq + m'} + \&\text{c.} = 0 ,$$

viz., the rationalised equation is an equation of the degree $2^{\nu-2}$ in q, and gives therefore $2^{\nu-2}$ values of q. And the ν-zomal curve thus breaks up into a system of $2^{\nu-2}$ curves each of the form $\Theta - q\Phi = 0$, that is, each of them in involution with the curves $\Theta = 0$, $\Phi = 0$. The equation in q may have a multiple root or roots, and the system of curves so contain repetitions of the same curve or curves; an instance of this (in relation to the trizomal curve) will present itself in the sequel; but I do not at present stop to consider the question.

26. A more important case is when the zomal curves are each of them in involution with the same two given curves, one of them of the order r, the other of an inferior order. Let $\Theta = 0$ be a curve of the order r, $\Phi = 0$ a curve of an inferior order $r - s$; $L = 0$, $M = 0$, &c., curves of the order s; then the case in question is when the zomal curves are of the form $\Theta + L\Phi = 0$, $\Theta + M\Phi = 0$, &c., the equation of the ν-zomal is

$$\sqrt{l(\Theta + L\Phi)} + \sqrt{m(\Theta + M\Phi)} + \&\text{c.} = 0,$$

where l, m, &c., are constants. This is the most convenient form for the equation, and by considering the functions L, M, &c. as containing implicitly the factors l^{-1}, m^{-1}, &c. respectively, we may take it to include the form $\sqrt{l\Theta + L\Phi} + \sqrt{m\Theta + M\Phi} + \&\text{c.} = 0$, which last has the advantage of being immediately applicable to the case where any one or more of the constants l, m, &c. may be $= 0$.

27. In the case now under consideration we have the $r(r - s)$ points of intersection of the curves $\Theta = 0$, $\Phi = 0$ as common points of all the zomals. Hence, putting in the foregoing formula $k = r(r - s)$, we have a ν-zomal curve of the order $2^{\nu-2}r$, having with each zomal $2^{\nu-2}rs$ contacts, or with all the zomals $2^{\nu-2}rs\nu$ contacts, having a node at each of the $2^{\nu-4}rs$ intersections (not being common points $\Theta = 0$, $\Phi = 0$) of each pair of complementary parazomals; that is, together $2^{\nu-4}(2^{\nu-1} - \nu - 1)rs$ nodes, and having, besides, at each of the

$r(r-s)$ common points, a $2^{\nu-2}$-tuple point, counting as $2^{2\nu-5}-2^{\nu-3}$ dps., together as $(2^{2\nu-5}-2^{\nu-3})\,r(r-s)$ dps.; whence, taking account of the nodes, the total number of dps. is $=2^{\nu-4}r\big[(2^{\nu-1}-2)r-(\nu-1)s\big]$.

Depression of Order of the ν-zomal Curve from the Ideal Factor of a Branch or Branches—Art. Nos. 28 to 37.

28. In the case of the $r(r-s)$ common points as thus far considered, the order of the ν-zomal curve has remained throughout $=2^{\nu-2}r$, but the order admits of depression, viz., the constants l, m, &c., and those of the functions L, M, &c., may be such that the norm contains the factor Φ^{ω}; the ν-zomal curve then contains as part of itself $(\Phi^{\omega}=0)$ the curve $\Phi=0$ taken ω times, and this being so, if we discard the factor in question, and consider the residual curve as being the ν-zomal, the order of the ν-zomal will be $=2^{\nu-2}r-\omega(r-s)$.

29. To explain how such a factor Φ^{ω} presents itself, consider the polyzome $\sqrt{l(\Theta+L\Phi)}$ + &c., or, what is the same thing, $\sqrt{l}\sqrt{\Theta+L\Phi}$ + &c., belonging to any particular branch of the curve, we may, it is clear, take $\sqrt{\Theta+L\Phi}$, &c. each in a fixed signification as equivalent to $\sqrt{\Theta+L\Phi}$, &c., respectively, and the particular branch will then be determined by means of the significations attached to $\sqrt{l}$, $\sqrt{m}$, &c. Expanding the several radicals, the polyzome is

$$\sqrt{l}\left\{\sqrt{\Theta}+\frac{1}{2}L\frac{\Phi}{\sqrt{\Theta}}-\frac{1}{8}L^2\frac{\Phi^2}{\Theta\sqrt{\Theta}}+\text{\&c.},\right\}+\text{\&c.};$$

or, what is the same thing, it is

$$\sqrt{\Theta}\left(\sqrt{l}+\text{\&c.}\right)+\frac{1}{2}\frac{\Phi}{\sqrt{\Theta}}\left(L\sqrt{l}+\text{\&c.}\right)-\frac{1}{8}\frac{\Phi^2}{\Theta\sqrt{\Theta}}\left(L^2\sqrt{l}+\text{\&c.}\right)+\text{\&c.}$$

which expansion may contain the factor Φ, or a higher power of Φ. For instance, if we have $\sqrt{l}$ + &c. $=0$, the expansion will then contain the factor Φ; and if we also have $L\sqrt{l}$ + &c. $=0$ (observe this implies as many equations as there are asyzygetic terms in the whole series of functions L, M, &c.; thus, if L, M, &c., are each of them of the form $aP+bQ+cR$, with the same values of P, Q, R, but with different values of the co-efficients a, b, c, then it implies the three equations $a\sqrt{l}$ + &c. $=0$, $b\sqrt{l}$ + &c. $=0$, $c\sqrt{l}$ + &c. $=0$; and so in other cases), if I say $L\sqrt{l}$ + &c. be also $=0$, then the expansion will contain the factor Φ^2, and so on; the most general supposition being, that the expansion contains as factor a certain power Φ^{α} of Φ. Imagine each of the polyzomes expanded in this manner, and let certain of the expansions contain the factors Φ^{α}, Φ^{β}, &c., respectively. The produce of the expansions is identically equal to the product of the unexpanded polyzomes—that is, it is equal to the Norm; hence, if $\alpha+\beta+$ &c. $=\omega$, the Norm will contain the factor Φ^{ω}.

30. It has been mentioned that the form $\sqrt{l(\Theta + L\Phi)}$ is considered as including the form $\sqrt{l\Theta + L\Phi}$, that is, when $l = 0$, the form $\sqrt{L\Phi}$. If in the equation of the ν-zomal curve there is any such term—for instance, if the equation be $\sqrt{L\Phi} + \sqrt{m(\Theta + M\Phi)}$ + &c. $= 0$—the radical $\sqrt{L\Phi}$ contains the factor $\Phi^{\frac{1}{2}}$; but if L contains as factor an odd or an even power of Φ, then $\sqrt{L\Phi}$ will contain the factor Φ^{α} where α is either an integer, or an integer $+ \frac{1}{2}$. Consider the polyzome $\sqrt{L\Phi} + \sqrt{m(\Theta + M\Phi)}$ + &c., belonging to any particular branch of the curve; the radical $\sqrt{L\Phi}$ contains, as just mentioned, the factor Φ^{α}, and if the remaining terms $\sqrt{m(\Theta + M\Phi)}$ + &c., are such that the expansion contains as factor the same or any higher power of Φ, then the expansion of the polyzome $\sqrt{L\Phi} + \sqrt{m(\Theta + M\Phi)}$ + &c., belonging to the particular branch will contain the factor Φ^{α}; and similarly we may have branches containing the factors Φ^{α}, Φ^{β}, &c., whence, as before, if $\omega = \alpha + \beta$ + &c., the Norm will contain the factor Φ^{ω}; the only difference is, that now α, β, &c., instead of being of necessity all integers, are each of them an integer, or an integer $+ \frac{1}{2}$; of course, in the latter case the integer may be zero, or the index be $= \frac{1}{2}$. It is clear that ω must be an integer, and it is, in fact, easy to see that the fractional indices occur in pairs; for observe that α being fractional, the expansion of $\sqrt{m(\Theta + M\Phi)}$ + &c., will contain not Φ^{α}, but a higher power, $\Phi^{\alpha + q}$, where $\alpha + q$ is an integer; whence *each* of the polyzomes $\sqrt{L\Phi} \pm (\sqrt{m(\Theta + M\Phi)}$ + &c.) will contain the factor Φ^{α}.

31. Observe that in every case the factor Φ^{α} presents itself as a factor of the expansion of the polyzome corresponding to a particular branch of the curve; the polyzome itself does not contain the factor Φ^{α}, and we cannot in anywise say that the corresponding branch contains as factor the curve $\Phi^{\alpha} = 0$; but we may, with great propriety of expression, say that *the branch ideally contains the curve* $\Phi^{\alpha} = 0$; and this being so, the general theorem is, that if we have branches ideally containing the curves $\Phi^{\alpha} = 0$, $\Phi^{\beta} = 0$, &c. respectively, then the ν-zomal curve contains not ideally but actually the factor $\Phi^{\omega} = 0$ ($\omega = \alpha + \beta$ + &c.), the order of the ν-zomal being thus reduced from $2^{\nu-2}r$ to $2^{\nu-2}r - \omega(r - s)$; and conversely, that any such reduction in the order of the ν-zomal arises from factors $\Phi^{\alpha} = 0$, $\Phi^{\beta} = 0$, &c., ideally contained in the several branches of the ν-zomal.

32. It is worth while to explain the notion of an ideal factor somewhat more generally; an irrational function, taking the irrationalities thereof in a determinate manner, may be such that, as well the function itself as all its differential co-efficients up to the order $\alpha - 1$, vanish when a certain parameter Φ contained in the function is put $= 0$; this is only saying, in other words, that the function expanded in ascending powers of Φ contains no power lower than Φ^{α}; and, in this case, we say that the irrational function contains *ideally* the factor Φ^{α}. The rationalised expression, or Norm, in virtue of the irrational function (taken determinately as above) thus ideally containing Φ^{α}, will actually contain the factor

Φ^α; and if any other values of the irrational function contain respectively Φ^β, &c., then the Norm will contain the factor $\Phi^{\alpha+\beta+\&c.}$

33. A branch ideally containing $\Phi^\alpha = 0$ may for shortness be called integral or fractional, according as the index α is an integer or a fraction; by what precedes the fractional branches present themselves in pairs. If for a moment we consider integral branches only, then if the ν-zomal contain $\Phi = 0$, this can happen in one way only, there must be some one branch ideally containing $\Phi = 0$; but if the ν-zomal contain $\Phi^2 = 0$, then this may happen in two ways,—either there is a single branch ideally containing $\Phi^2 = 0$, or else there are two branches, each of them ideally containing $\Phi = 0$. And generally, if the ν-zomal contain $\Phi^\omega = 0$, then forming any partition $\omega = \alpha + \beta +$ &c. (the parts being integral), this may arise from there being branches ideally containing $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively. The like remarks apply to the case where we attend also to fractional branches,—thus, if the ν-zomal contain $\Phi = 0$, this may arise (not only, as above mentioned, from a branch ideally containing $\Phi = 0$, but also) from a pair of branches, each ideally containing $\Phi^{\frac{1}{2}} = 0$. And so in general, if the ν-zomal contain $\Phi^\omega = 0$, the partition $\omega = \alpha + \beta +$ &c. is to be made with the parts integral or fractional ($= \frac{1}{2}$ or integer $+ \frac{1}{2}$ as above), but with the fractional terms in pairs; and then the factor $\Phi^\omega = 0$ may arise from branches ideally containing $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively.

34. Any zomal, antizomal, or parazomal of a ν-zomal curve, $\sqrt{l(\Theta + L\Phi)} +$ &c. $= 0$, is a polyzomal curve (including in the term a monozomal curve) of the same form as the ν-zomal; and may in like manner contain $\Phi = 0$, or more generally, $\Phi^\omega = 0$, viz., if $\omega = \alpha + \beta +$ &c. be any partition of ω as above, this will be the case if the zomal, antizomal, or parazomal has branches ideally containing $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively. It is to be observed that if a zomal, antizomal, or parazomal contain $\Phi = 0$, or any higher power $\Phi^\omega = 0$, this does not in anywise imply that the zomal contains even $\Phi = 0$. But if (attending only to the most simple case) a zomal and its antizomal, or a pair of complementary parazomals, each contain $\Phi = 0$ inseparably (that is, through a single branch ideally containing $\Phi = 0$), then the ν-zomal will have two branches, each ideally containing $\Phi = 0$, and it will thus contain $\Phi^2 = 0$. In fact, if in the zomal and antizomal, or in the complementary parazomals, the branches which ideally contain $\Phi = 0$ are

$$\sqrt{l(\Theta + L\Phi)} + \&c. = 0,\ \sqrt{n(\Theta + N\Phi)} + \&c. = 0$$

respectively (for a zomal, the + &c. should be omitted, and the first equation be written $\sqrt{l(\Theta + L\Phi)} = 0$), then in the ν-zomal there will be the two branches

$$\left(\sqrt{l(\Theta + L\Phi)} + \&c.\right) \pm \left(\sqrt{n(\Theta + N\Phi)} + \&c.\right) = 0\,,$$

each ideally containing $\Phi = 0$.

Conversely, if a ν-zomal contain $\Phi^2 = 0$ by reason that it has two branches each ideally containing $\Phi = 0$, then either a zomal and its antizomal will each of them, or else a pair of complementary parazomals will each of them, inseparably contain $\Phi = 0$.

35. Reverting to the case of the ν-zomal curve

$$\sqrt{l(\Theta + L\Phi)} + \sqrt{m(\Theta + M\Phi)} + \&c. = 0\,,$$

which does not contain $\Phi = 0$, each of the $r(r - s)$ common points $\Theta = 0$, $\Phi = 0$ is a $2^{\nu-2}$-tuple point on the ν-zomal; each of these counts therefore for $2^{\nu-2}$ intersections of the ν-zomal with the curve $\Phi = 0$, and we have thus the complete number $2^{\nu-2}\, r(r - s)$ of intersections of the two curves, viz., the curve $\Phi = 0$ meets the ν-zomal in the $r(r - s)$ common points, each of them a $2^{\nu-2}$-tuple point on the ν-zomal, and in no other point.

36. But if the ν-zomal contains $\Phi^\omega = 0$, then each of the $r(r - s)$ common points is still a $2^{\nu-2}$-tuple point on the aggregate curve; the aggregate curve therefore passes $2^{\nu-2}$ times through each common point; but among these passages are included ω passages of the curve $\Phi = 0$ through the common point. The residual curve—say the ν-zomal—passes therefore only $2^{\nu-2} - \omega$ times through the common point; that is, each of the $r(r - s)$ common points is a $(2^{\nu-2} - \omega)$ tuple point on the ν-zomal. The curve $\Phi = 0$ meets the ν-zomal in $\{2^{\nu-2}\, r - \omega(r - s)\}\, (r - s)$ points, viz., these include the $r(r - s)$ common points, each of them a $(2^{\nu-2} - \omega)$ tuple point on the ν-zomal, and therefore counting together as $(2^{\nu-2} - \omega)\, r(r - s)$ intersections; there remain consequently $\omega s(r - s)$ other intersections of the curve $\Theta = 0$ with the ν-zomal.

37. In the case where the ν-zomal contains the factor $\Phi^\omega = 0$, then throughout excluding from consideration the $r(r - s)$ common points $\Theta = 0$, $\Phi = 0$, the *remaining* intersections of any zomal with its antizomal are points of contact of the zomal with the ν-zomal, and the *remaining* intersections of each pair of complementary parazomals are nodes of the ν-zomal, it being understood that if any zomal, antizomal, or parazomal contain a power of $\Phi = 0$, such powers of $\Phi = 0$ are to be discarded, and only the residual curves attended to. The number of contacts and of nodes may in any particular case be investigated without difficulty, and some instances will present themselves in the sequel, but on account of the different ways in which the factor $\Phi^\omega = 0$ may present itself, ideally in a single branch, or in several branches, and the consequent occurrence in the latter case of powers of $\Phi = 0$ in certain of the zomals, antizomals, or parazomals, the cases to be considered would be very numerous, and there is no reason to believe that the results could be presented in any moderately concise form; I therefore abstain from entering on the question.

On the Trizomal Curve and the Tetrazomal Curve—Art. Nos. 38 and 39.

38. The trizomal curve

$$\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$$

has for its rationalised form of equation

$$U^2 + V^2 + W^2 - 2VW - 2WU - 2UV = 0;$$

or as this may also be written,

$$(1, 1, 1, -1, -1, -1)(U, V, W)^2 = 0;$$

and we may from this rational equation verify the general results applicable to the case in hand, viz., that the trizomal is a curve of the order $2r$, and that

$U = 0$, at each of its r^2 intersections with $V - W = 0$
$V = 0$, „ „ $W - U = 0$
$W = 0$, „ „ $U - V = 0$

respectively touch the trizomal. There are not, in general, any nodes or cusps, and the order being $= 2r$, the class is $= 2r(2r - 1)$.

39. The tetrazomal curve

$$\sqrt{U} + \sqrt{V} + \sqrt{W} + \sqrt{T} = 0$$

has for its rationalised form of equation

$$(U^2 + V^2 + W^2 + T^2 - 2UV - 2UW - 2UT - 2VW - 2VT - 2WT)^2 - 64UVWT = 0,$$

and we may hereby verify the fundamental properties, viz., that the tetrazomal is a curve of the order $4r$, touched by each of the zomals $U = 0$, $V = 0$, $W = 0$, $T = 0$ in $2r^2$ points, viz. by $U = 0$ at its intersections with $\sqrt{U} + \sqrt{W} + \sqrt{T} = 0$, that is, $V^2 + W^2 + T^2 - 2VW - 2VT - 2WT = 0$; and the like as regards the other zomals), and having $3r^2$ nodes, viz., these are the intersections of $(\sqrt{U} + \sqrt{V} = 0, \sqrt{W} + \sqrt{T} = 0)$, $(\sqrt{U} + \sqrt{W} = 0, \sqrt{V} + \sqrt{T} = 0)$, $(\sqrt{U} + \sqrt{T} = 0, \sqrt{V} + \sqrt{W} = 0)$, or, what is the same thing, the intersections of $(U - V = 0, W - T = 0)$, $(U - W = 0, V - T = 0)$, $(U - T = 0, V - W = 0)$. There are not in general any cusps, and the class is thus $= 4r(4r - 1) - 6r^2$, $= 10r^2 - 4r$.

On the Intersection of two ν-Zomals having the same Zomal Curves—Art. Nos. 40 and 41.

40. Without going into any detail, I may notice the question of the intersection of two ν-zomals which have the same zomal curves—say the two trizomals $\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$, $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$, or two similarly related

tetrazomals. For the trizomals, writing the equations under the form

$$\sqrt{U}+\sqrt{V}+\sqrt{W}=0,\quad \sqrt{l}\sqrt{U}+\sqrt{m}\sqrt{V}+\sqrt{n}\sqrt{W}=0\,,$$

then, when these equations are considered as existing simultaneously, we may, without loss of generality, attribute to the radicals $\sqrt{U}$, $\sqrt{V}$, $\sqrt{W}$, the same values in the two equations respectively; but doing so, we must in the second equation successively attribute to all but one of the radicals $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$, each of its two opposite values. For the intersections of the two curves we have thus

$$\sqrt{U}:\sqrt{V}:\sqrt{W}=\sqrt{m}-\sqrt{n}:\sqrt{n}-\sqrt{l}:\sqrt{l}-\sqrt{m}\,,$$

viz., this is one of a system of four equations, obtained from it by changes of sign, say in the radicals $\sqrt{m}$ and $\sqrt{n}$. Each of the four equations gives a set of r^2 points; we have thus the complete number, $=4r^2$, of the points of intersection of the two curves.

41. But take, in like manner, two tetrazomal curves; writing their equations in the form

$$\sqrt{U}+\sqrt{V}+\sqrt{W}+\sqrt{T}=0\,,$$
$$\sqrt{l}\sqrt{U}+\sqrt{m}\sqrt{V}+\sqrt{n}\sqrt{W}+\sqrt{p}\sqrt{T}=0\,,$$

then $\sqrt{U}$, $\sqrt{V}$, $\sqrt{W}$, $\sqrt{T}$ may be considered as having the same values in the two equations respectively, but we must in the second equation attribute successively, say to $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$, each of their two opposite values. For the intersections of the two curves we have

$$\begin{aligned} &(\sqrt{m}-\sqrt{l})\sqrt{V}+(\sqrt{n}-\sqrt{l})\sqrt{W}+(\sqrt{p}-\sqrt{l})\sqrt{T}=0\\ (\sqrt{l}-\sqrt{m})\sqrt{U}\quad &\qquad\qquad\qquad\quad +(\sqrt{n}-\sqrt{m})\sqrt{W}+(\sqrt{p}-\sqrt{m})\sqrt{T}=0 \end{aligned}$$

viz., this is one of a system of eight similar pairs of equations, obtained therefrom by changes of sign of the radicals $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$. The equations represent each of them a trizomal curve, of the order $2r$; the two curves intersect therefore in $4r^2$ points, and if each of these was a point of intersection of the two tetrazomals, we should have in all $8\times 4r^2=32r^2$ intersections. But the tetrazomals are each of them a curve of the order $4r$, and they intersect therefore in only $16r^2$ points. The explanation is, that not all the $4r^2$ points, but only $2r^2$ of them are intersections of the tetrazomals. In fact, to find *all* the intersections of the two trizomals, it is necessary in their two equations to attribute opposite signs to one of the radicals $\sqrt{W}$, $\sqrt{T}$; we obtain $2r^2$ intersections from the equations as they stand, the remaining $2r^2$ intersections from the two equations after we have in the second equation reversed the sign, say of $\sqrt{T}$. Now, from the two equations as they stand we can pass back to the two tetrazomal equations, and the first mentioned $2r^2$ points are thus points of intersection of the two tetrazomal curves — from the two equations after such reversal of the sign of $\sqrt{T}$, we cannot pass back to the two tetrazomal equations, and the last-mentioned $2r^2$ points are thus not points of intersection of the two tetrazomal curves. The number of intersections of the two curves is thus $8 \times 2r^2, = 16r^2$, as it should be.

The Theorem of the Decomposition of a Tetrazomal Curve—Art. Nos. 42 to 45.

42. I consider the tetrazomal curve

$$\sqrt{lU} + \sqrt{mV} + \sqrt{nW} + \sqrt{pT} = 0\,,$$

where the zomal curves are in involution,—that is, where we have an identical relation,

$$\mathrm{a}U + \mathrm{b}V + \mathrm{c}W + \mathrm{d}T = 0\,;$$

and I proceed to show that if l, m, n, p, satisfy the relation

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0\,,$$

the curve breaks up into two trizomals. In fact, writing the equation under the form

$$(\sqrt{lU} + \sqrt{mV} + \sqrt{nW})^2 - pT = 0\,.$$

and substituting for T its value, in terms of U, V, W, this is

$$\begin{aligned}&(l\mathrm{d} + p\mathrm{a})U + (m\mathrm{d} + p\mathrm{b})V + (n\mathrm{d} + p\mathrm{c})W \\ &+ 2\sqrt{mn\mathrm{d}}\,\sqrt{VW} + 2\sqrt{nl\mathrm{d}}\,\sqrt{WU} + 2\sqrt{lm\mathrm{d}}\,\sqrt{UV} = 0\,:\end{aligned}$$

or, considering the left-hand side as a quadric function of $(\sqrt{U}, \sqrt{V}, \sqrt{W})$, the condition for its breaking up into factors is

$$\begin{vmatrix} l\mathrm{d} + p\mathrm{a}, & \mathrm{d}\sqrt{lm}, & \mathrm{d}\sqrt{ln} \\ \mathrm{d}\sqrt{ml}, & m\mathrm{d} + p\mathrm{b}, & \mathrm{d}\sqrt{mn} \\ \mathrm{d}\sqrt{nl}, & \mathrm{d}\sqrt{nm}, & n\mathrm{d} + p\mathrm{c} \end{vmatrix} = 0\,.$$

that is

$$p^2(l\mathrm{bcd} + m\mathrm{cda} + n\mathrm{dab} + p\mathrm{abc}) = 0\,,$$

or finally, the condition is

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0\,.$$

43. Multiplying by $l\mathrm{d} + p\mathrm{a}$, and observing that in virtue of the relation we have

$$(l\mathrm{d} + p\mathrm{a})\,(m\mathrm{d} + p\mathrm{b}) = lm\mathrm{d}^2 - \frac{\mathrm{abd}}{\mathrm{c}}pn\,,$$

$$(l\mathrm{d} + p\mathrm{a})\,(n\mathrm{d} + p\mathrm{c}) = ln\mathrm{d}^2 - \frac{\mathrm{acd}}{\mathrm{b}}pm\,,$$

the equation becomes

$$\left((l\mathrm{d} + p\mathrm{a})\,\sqrt{U} + \mathrm{d}\sqrt{lm}\,\sqrt{V} + \mathrm{d}\sqrt{ln}\,\sqrt{W}\right)^2 = \frac{\mathrm{ad}}{\mathrm{bc}}\,p\left(\mathrm{b}\sqrt{n}\,\sqrt{V} - \mathrm{c}\sqrt{m}\,\sqrt{W}\right)^2$$

or as this is more conveniently written

$$\left(\left(\sqrt{\;}_{1} + \frac{\mathrm{a}p}{\mathrm{d}\sqrt{l}}\right)\sqrt{U} + \sqrt{mV} + \sqrt{nW}\right)^2 = \frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}\left(\mathrm{b}\sqrt{nV} - \mathrm{c}\sqrt{mV}\right)^2,$$

an equation breaking up into two equations, which may be represented by

$$\sqrt{l_1 U} + \sqrt{m_1 V} + \sqrt{n_1 W} = 0,\ \ \sqrt{l_2 U} + \sqrt{m_2 V} + \sqrt{n_2 W} = 0\,,$$

where

$$\begin{aligned} \sqrt{l_1} &= \sqrt{l} + \frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}} &,\quad \sqrt{l_2} &= \sqrt{l} + \frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}} \\ \sqrt{m_1} &= \sqrt{m} - \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{b}\sqrt{n} &,\quad \sqrt{m_2} &= \sqrt{m} + \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{b}\sqrt{n} \\ \sqrt{n_1} &= \sqrt{n} + \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{c}\sqrt{m} &,\quad \sqrt{n_2} &= \sqrt{n} - \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{c}\sqrt{m}, \end{aligned}$$

where, in the expressions for $\sqrt{l}$, &c., the signs of the radicals

$$\sqrt{l},\ \sqrt{m},\ \sqrt{n},\ \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,,$$

may be taken determinately in any way whatever at pleasure; the only effect of an alteration of sign would in some cases be to interchange the values of $(\sqrt{l_1}, \sqrt{m_1}, \sqrt{n_1})$ with those of $(\sqrt{l_2}, \sqrt{m_2}, \sqrt{n_2})$. The tetrazomal curve thus breaks up into two trizomals.

44. It is to be noticed that we have

$$\begin{aligned} \frac{l_1}{\mathrm{a}} + \frac{m_1}{\mathrm{b}} + \frac{n_1}{\mathrm{c}} = \frac{l}{\mathrm{a}} &+ \frac{\mathrm{a}p^2}{\mathrm{d}^2 l} + 2\frac{p}{\mathrm{d}} \\ &+ \frac{m}{\mathrm{b}} + \frac{\mathrm{a}}{\mathrm{cd}}\frac{np}{l} \\ &+ \frac{n}{\mathrm{c}} + \frac{\mathrm{a}}{\mathrm{bd}}\frac{mp}{l} \\ &= \left(l + \frac{\mathrm{a}p}{\mathrm{d}l}\right)\left(\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}}\right); \end{aligned}$$

that is

$$\frac{l_1}{\mathrm{a}} + \frac{m_1}{\mathrm{b}} + \frac{n_1}{\mathrm{c}} = 0\,.$$

And similarly we have

$$\frac{l_2}{\mathrm{a}} + \frac{m_2}{\mathrm{b}} + \frac{n_2}{\mathrm{c}} = 0\,.$$

The meaning is, that, taking the trizomal curve $\sqrt{l_1 U} + \sqrt{m_1 V} + \sqrt{n_1 W} = 0$, this regarded as a tetrazomal curve, $\sqrt{l_1 U} + \sqrt{m_1 V} + \sqrt{n_1 W} + \sqrt{0T} = 0$, satisfies the condition $\frac{l_1}{\mathrm{a}} + \frac{m_1}{\mathrm{b}} + \frac{n_1}{\mathrm{c}} + \frac{0}{\mathrm{d}} = 0$; and the like as to the trizomal curve $\sqrt{l_2 U} + \sqrt{m_2 V} + \sqrt{n_2 W} = 0$.

45. The equation by which the decomposition was effected is, it is clear, one of twelve equivalent equations; four of these are

$$\left(\sqrt{l}+\frac{\mathrm{a}p}{\mathrm{d}\sqrt{l}},\ \sqrt{m},\ \sqrt{n},\ 0\right)\left(\sqrt{U},\ \sqrt{V},\ \sqrt{W},\ \sqrt{T}\right)^2 = \frac{\mathrm{a}}{\mathrm{bcd}}\,\frac{p}{l}\left(\mathrm{b}\sqrt{nW}-\mathrm{c}\sqrt{mW}\right)^2$$

$$\left(0,\ \sqrt{m}+\frac{\mathrm{b}l}{\mathrm{a}\sqrt{m}},\ \sqrt{n},\ \sqrt{p}\right)\left(\quad ,, \quad\right)^2 = \frac{\mathrm{a}}{\mathrm{cda}}\,\frac{l}{m}\left(\mathrm{c}\sqrt{pW}-\mathrm{d}\sqrt{nT}\right)^2$$

$$\left(\sqrt{l},\ 0,\ \sqrt{n}+\frac{\mathrm{c}m}{\mathrm{b}\sqrt{n}},\ \sqrt{p}\right)\left(\quad ,, \quad\right)^2 = \frac{\mathrm{c}}{\mathrm{dab}}\,\frac{m}{n}\left(\mathrm{d}\sqrt{lT}-\mathrm{a}\sqrt{pU}\right)^2$$

$$\left(\sqrt{l},\ \sqrt{m},\ 0,\ \sqrt{p}+\frac{\mathrm{d}n}{\mathrm{c}\sqrt{p}}\right)\left(\quad ,, \quad\right)^2 = \frac{\mathrm{d}}{\mathrm{abc}}\,\frac{n}{p}\left(\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV}\right)^2$$

and the others may be deduced from these by a cyclical permutation of $(U,\ V,\ W)$, (a, b, c), $(l,\ m,\ n)$, leaving T, d, p unaltered.

Application to the Trizomal; the Theorem of the Variable Zomal—Art. Nos. 46 to 51.

46. I take the last equation written under the form

$$(\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV})^2 = \frac{\mathrm{abc}}{\mathrm{d}n}\left(\sqrt{lpU}+\sqrt{mpV}+\left(p+\frac{\mathrm{d}n}{\mathrm{c}}\right)\sqrt{T}\right)^2,$$

which, putting therein $p=0$, is

$$(\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV})^2 = \frac{\mathrm{abd}}{\mathrm{c}}\,nT,$$

which is in fact the trizomal curve,

$$\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV}+\sqrt{\frac{\mathrm{abd}}{\mathrm{c}}\,nT}=0,$$

viz., the trizomal curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$,—if a, b, c be any quantities connected by the equation

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0,$$

(the ratios a, b, c thus involving a single arbitrary parameter); and if we take T a function such that $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W+\mathrm{d}T=0$; that is, $T=0$, any one of the series of curves $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$, in involution with the given curves $U = 0$, $V = 0$, $W = 0$,—has its equation expressible in the form

$$\mathrm{a}\sqrt{mU} - \mathrm{b}\sqrt{lV} + \sqrt{\frac{\mathrm{abd}}{\mathrm{c}}nT} = 0\,;$$

that is, we have the curve $T = 0$ (the equation whereof contains a variable parameter) as a zomal of the given trizomal curve $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$; and we have thus from the theorem of the decomposition of a tetrazomal deduced the theorem of the variable zomal of a trizomal. The analytical investigation is somewhat simplified by assuming $p = 0$ *ab initio*, and it may be as well to repeat it in this form.

47. Starting, then, with the trizomal curve

$$\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0\,,$$

and writing

$$\mathrm{a}U + \mathrm{b}V + \mathrm{c}W + \mathrm{d}T = 0$$

as the definition of T, the coefficients being connected by

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0\,,$$

the equation gives

$$lU + mV + 2\sqrt{lmUV} - nW = 0\,;$$

or substituting in this equation for W its value in terms of U, V, T, we have

$$(\mathrm{a}n + \mathrm{c}l)\,U + (\mathrm{b}n + \mathrm{c}m)\,V + 2\mathrm{c}\sqrt{lmUV} + \mathrm{d}nT = 0\,,$$

which by the given relation between a, b, c, is converted into

$$-\frac{\mathrm{ac}}{\mathrm{b}}mU - \frac{\mathrm{bc}}{\mathrm{a}}lV + 2\mathrm{c}\sqrt{lmUV} + \mathrm{d}nT = 0\,;$$

that is

$$\mathrm{a}^2mU + \mathrm{b}^2lV - 2\mathrm{ab}\sqrt{lmUV} = \frac{\mathrm{abd}}{\mathrm{c}}nT\,,$$

viz., this is

$$(\mathrm{a}\sqrt{mU} - \mathrm{b}\sqrt{lV})^2 = \frac{\mathrm{abd}}{\mathrm{c}}nT\,,$$

or finally

$$\mathrm{a}\sqrt{mU} - \mathrm{b}\sqrt{lV} + \sqrt{\frac{\mathrm{abd}}{\mathrm{c}}nT} = 0\,.$$

48. The result just obtained of course implies that when as above

$$\mathrm{a}U + \mathrm{b}V + \mathrm{c}W + \mathrm{d}T = 0,\ \frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0\,,$$

the trizomal curve $\sqrt{lU} + \sqrt{mN} + \sqrt{nW} = 0$ can be expressed by means of any three of the four zomals U, V, W, T, and we may at once write down the four forms

$$\left(\begin{array}{cccc} \cdot & \sqrt{\frac{n}{c^2}}, & -\sqrt{\frac{m}{b^2}}, & -\sqrt{\frac{ld}{abc}} \\ -\sqrt{\frac{n}{c^2}}, & \cdot & \sqrt{\frac{l}{a^2}}, & -\sqrt{\frac{md}{abc}} \\ \sqrt{\frac{m}{b^2}}, & -\sqrt{\frac{l}{a^2}} & \cdot & -\sqrt{\frac{nd}{abc}} \\ \sqrt{\frac{ld}{abc}}, & \sqrt{\frac{md}{abc}}, & \sqrt{\frac{nd}{abc}}, & \cdot \end{array}\right)(\sqrt{U}, \sqrt{V}, \sqrt{W}, \sqrt{T})=0,$$

the last of which is the original equation $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$. It may be added that if the first equation be represented by $\sqrt{m_1 V} + \sqrt{n_1 W} + \sqrt{p_1 T} = 0$, —that is, if we have

$$\sqrt{m_1} = \sqrt{\frac{n}{c^2}},\quad \sqrt{n_1} = -\sqrt{\frac{m}{b^2}},\quad \sqrt{p_1} = \sqrt{\frac{ld}{abc}},$$

and therefore,

$$\frac{m_1}{b} + \frac{n_1}{c} + \frac{p_1}{d} = \frac{l}{bc}\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right), = 0;$$

or if the second equation be represented by $\sqrt{l_2 U} + \sqrt{n_2 W} + \sqrt{p_2 T} = 0$,—that is, if we have

$$\sqrt{l_2} = -\sqrt{\frac{n}{c^2}},\quad \sqrt{n_2} = \sqrt{\frac{l}{a^2}},\quad \sqrt{p_2} = \sqrt{\frac{md}{abc}},$$

and therefore

$$\frac{l_2}{a} + \frac{n_2}{c} + \frac{p_2}{d} = 0;$$

or if the third equation be represented by $\sqrt{l_3 U} + \sqrt{m_3 V} + \sqrt{p_3 T} = 0$,—that is, if we have

$$\sqrt{l_3} = \sqrt{\frac{m}{b^2}},\quad \sqrt{m_3} = -\sqrt{\frac{l}{a^2}},\quad \sqrt{p_3} = \sqrt{\frac{md}{abc}},$$

and therefore

$$\frac{l_3}{a} + \frac{m_3}{b} + \frac{p_3}{d} = 0,$$

then the equation of the trizomal may also be expressed in the forms—

$$\left(\begin{array}{cccc} \cdot\ , & \sqrt{m_1}\ , & \sqrt{n_1}\ , & \sqrt{p_1} \\ -\sqrt{m_1}, & \cdot\ , & \sqrt{\frac{p_1 bc}{ad}}, & -\sqrt{\frac{n_1 bd}{ac}} \\ -\sqrt{n_1}\ , & -\sqrt{\frac{p_1 bc}{ad}}, & \cdot\ , & \sqrt{\frac{m_1 cd}{ab}} \\ -\sqrt{p_1}\ , & \sqrt{\frac{n_1 bc}{ad}}, & -\sqrt{\frac{m_1 cd}{ab}}, & \end{array}\right)(\sqrt{U}, \sqrt{V}, \sqrt{W}, \sqrt{T})=0$$

$$\begin{pmatrix} \cdot\;, & -\sqrt{l_2}\;, & -\sqrt{\frac{p_2\mathrm{ac}}{\mathrm{bd}}}, & \sqrt{\frac{n_2\mathrm{ad}}{\mathrm{bc}}} \\ \sqrt{l_2}\;, & \cdot\;, & \sqrt{n_2}\;, & \sqrt{p_2} \\ \sqrt{\frac{p_2\mathrm{ac}}{\mathrm{bc}}}, & -\sqrt{n_2}, & \cdot\;, & -\sqrt{\frac{l_2\mathrm{cd}}{\mathrm{ab}}} \\ -\sqrt{\frac{n_2\mathrm{ad}}{\mathrm{bc}}}, & -\sqrt{p_2}\;, & -\sqrt{\frac{l_2\mathrm{cd}}{\mathrm{ab}}}\;, & \cdot \end{pmatrix}\left(\sqrt{U}, \sqrt{V}, \sqrt{W}, \sqrt{T}\right) = 0$$

and

$$\begin{pmatrix} \cdot\;, & \sqrt{\frac{p_3\mathrm{ab}}{\mathrm{cd}}}, & -\sqrt{l_3}\;, & -\sqrt{\frac{m_3\mathrm{ad}}{\mathrm{bc}}} \\ -\sqrt{\frac{p_3\mathrm{ab}}{\mathrm{cd}}}\;, & \cdot\;, & -\sqrt{m_3}, & -\sqrt{\frac{l_3\mathrm{bd}}{\mathrm{ac}}} \\ \sqrt{l_4}\;, & \sqrt{m_3}\;, & \cdot\;, & \sqrt{p_3} \\ \sqrt{\frac{m_3\mathrm{ad}}{\mathrm{bc}}}, & -\sqrt{\frac{l_3\mathrm{b}p}{\mathrm{ac}}}\;, & -\sqrt{p_3}\;, & \cdot \end{pmatrix}\left(\sqrt{U}, \sqrt{V}, \sqrt{W}, \sqrt{T}\right) = 0$$

49. These equations may, however, be expressed in a much more elegant form. Write

$$\mathrm{a}' = \frac{a}{(\beta\gamma\delta)},\quad \mathrm{b}' = -\frac{b}{(\gamma\delta\alpha)},\quad \mathrm{c}' = \frac{c}{(\delta\alpha\beta)},\quad \mathrm{d}' = \frac{-d}{(\alpha\beta\gamma)},$$

where, for shortness, $(\beta\gamma\delta) = (\beta-\gamma)(\gamma-\delta)(\delta-\beta)$, &c.; (α, β, γ) being arbitrary quantities: or, what is the same thing,

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = \mathrm{a}'(\beta\gamma\delta) : -\mathrm{b}'(\gamma\delta\alpha) : \mathrm{c}'(\delta\alpha\beta) : -\mathrm{d}'(\alpha\beta\gamma)\,.$$

Assume

$$l : m : n = \varrho\mathrm{a}'(\beta-\gamma)^2 : \sigma\mathrm{b}'(\gamma-\alpha)^2 : \tau\mathrm{c}'(\alpha-\beta)^2\,;$$

then the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$ takes the form

$$\varrho(\beta-\gamma)(\alpha-\delta) + \sigma(\gamma-\alpha)(\beta-\delta) + \tau(\alpha-\beta)(\gamma-\delta)\,,$$

and the four forms of the equation are found to be

$$\begin{pmatrix} \cdot\;, & \sqrt{\tau}(\delta-\gamma), & \sqrt{\sigma}(\beta-\delta), & \sqrt{\varrho}(\gamma-\beta) \\ \sqrt{\tau}(\gamma-\delta), & \cdot\;, & \sqrt{\varrho}(\delta-\alpha), & \sqrt{\sigma}(\alpha-\gamma \\ \sqrt{\sigma}(\delta-\beta), & \sqrt{\varrho}(\alpha-\delta)\,, & \cdot\;, & \sqrt{\tau}(\beta-\alpha) \\ \sqrt{\varrho}(\beta-\gamma), & \sqrt{\sigma}(\gamma-\alpha), & \sqrt{\tau}(\alpha-\beta) & \cdot \end{pmatrix}\left(\sqrt{\mathrm{a}'U}, \sqrt{\mathrm{b}'V}, \sqrt{\mathrm{c}'W}, \sqrt{\mathrm{d}'T}\right) = 0$$

viz., these are the equivalent forms of the original equation assumed to be $(\beta-\gamma)\sqrt{\rho\mathrm{a}'U} + (\gamma-\alpha)\sqrt{\sigma\mathrm{b}'V} + (\alpha-\beta)\sqrt{\tau\mathrm{c}'W} = 0$.

50. I remark that the theorem of the variable zomal may be obtained as a transformation theorem—viz., comparing the equation $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$ with the equation $\sqrt{lx} + \sqrt{my} + \sqrt{nz} = 0$; this last belongs to a conic touched by the three lines $x = 0$, $y = 0$, $z = 0$; the equation of the same conic must, it is clear, be expressible in a similar form by means of any other three tangents

thereof, but the equation of any tangent of the conic is $\mathrm{a}x + \mathrm{b}y + \mathrm{c}z = 0$, where a, b, c are any quantities satisfying the condition $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$; whence, writing $\mathrm{a}x + \mathrm{b}y + \mathrm{c}z + \mathrm{d}w = 0$, we may introduce $w = 0$ along with any two of the original zomals $x = 0$, $y = 0$, $z = 0$, or, instead of them, any three functions of the form w; and then the mere change of x, y, z, w into U, V, W, T gives the theorem. But it is as easy to conduct the analysis with (U, V, W, T) as with (x, y, z, w), and, so conducted, it is really the same analysis as that whereby the theorem is established *ante*, No. 47.

51. It is worth while to exhibit the equation of the curve

$$\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0,$$

in a form containing three new zomals. Observe that the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$ is satisfied by $\mathrm{a} = l\phi\chi$, $\mathrm{b} = m\chi\theta$, $\mathrm{c} = n\theta\phi$, if only $\theta + \phi + \chi = 0$; or say, if $\theta = a' - a''$, $\phi = a'' - a$, $\chi = a - a''$. The equation

$$\begin{aligned} &\lambda \sqrt{(a-a')(a-a'')lU + (a'-a'')(a'-a)mV + (a''-a)(a''-a')nW} \\ +\, &\mu \sqrt{(b-b')(b-b'')lU + (b'-b'')(b'-b)mV + (b''-b)(b''-b')nW} \\ +\, &\nu \sqrt{(c-c')(c-c'')lU + (c'-c'')(c'-c)mV + (c''-c)(c''-c')nW} = 0 \end{aligned}$$

is consequently an equation involving three zomals of the proper form; and we can determine λ, μ, ν in suchwise as to identify this with the original equation $\sqrt{lU} + \sqrt{mV} + \sqrt{nW}$, viz., writing successively $U = 0$, $V = 0$, $W = 0$, we find

$$\begin{aligned} (a'-a'')\lambda + (b'-b'')\mu + (c'-c'')\nu &= 0\,, \\ (a''-a)\lambda + (b''-b)\mu + (c''-c)\nu &= 0\,, \\ (a-a')\lambda + (b-b')\mu + (c-c'\;\nu &= 0\,, \end{aligned}$$

equations which are, as they should be, equivalent to two equations only, and which give

$$\lambda : \mu : \nu = \begin{vmatrix} 1, 1, 1 \\ b, b', b'' \\ c, c', c'' \end{vmatrix} : \begin{vmatrix} 1, 1, 1 \\ c, c', c'' \\ a, a', a'' \end{vmatrix} : \begin{vmatrix} 1, 1, 1 \\ a, a', a'' \\ b, b', b'' \end{vmatrix},$$

and the equation, with these values of λ, μ, ν substituted therein, is in fact the equation of the trizomal curve $\sqrt{W} + \sqrt{mV} + \sqrt{nW} = 0$ in terms of three new zomals. It is easy to return to the forms involving one new zomal and any two of the original three zomals.

Remark as to the Tetrazomal Curve—Art. No. 52.

52. I return for a moment to the case of the tetrazomal curve, in order to show that there is not, in regard to it in general, any theorem such as that of the variable zomal. Considering the form $\sqrt{lx} + \sqrt{my} + \sqrt{nz} + \sqrt{pw} = 0$ (the co-

ordinates x, y, z, w are of course connected by a linear equation, but nothing turns upon this), the curve is here a quartic touched twice by each of the lines $x = 0$, $y = 0$, $z = 0$, $w = 0$ (viz., each of these is a double tangent of the curve), and having besides the three nodes $(x = y, z = w)$, $(x = z, y = w)$, $(x = w, y = z)$. But a quartic curve with three nodes, or trinodal quartic, has only four double tangents—that is, besides the lines $x = 0$, $y = 0$, $z = 0$, $w = 0$, there is no line $\alpha x + \beta y + \gamma z + \delta w = 0$ which is a double tangent of the curve; and writing U, V, W, T in place of x, y, z, w, then if U, V, W, T are connected by a linear equation (and, *à fortiori*, if they are not so connected), there is not any curve $\alpha U + \beta V + \gamma W + \delta T = 0$ which is related to the curve in the same way with the lines $U = 0$, $V = 0$, $W = 0$, $T = 0$; or say there is not (besides the curves $U = 0$, $V = 0$, $W = 0$, $T = 0$), any other zomal $\alpha U + \beta V + \gamma W + \delta T = 0$, of the tetrazomal curve. The proof does not show that for special forms of U, V, W, T there may not be zomals, not of the above form $\alpha U + \beta V + \gamma W + \delta T = 0$, but belonging to a separate system. An instance of this will be mentioned in the sequel.

The Theorem of the Variable Zomal of a Trizomal Curve resumed—Art. Nos. 53 to 56.

53. I resume the theorem of the variable zomal of the trizomal curve $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$. The variable zomal $T = 0$ is the curve $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$, where a, b, c are connected by the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$; that is, it belongs to a single series of curves selected in a certain manner out of the double series $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$ (a double series, as containing the *two* variable parameters a : b : c). These are the whole series of curves in involution with the given curves $U = 0$, $V = 0$, $W = 0$, or being such that the Jacobian of any three of them is identical with the Jacobian of the three given curves; in particular, the Jacobian of any one of the curves $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$, and of two of the three given curves, is identical with the Jacobian of the three given curves. I call to mind that, by the Jacobian of the curves $U = 0$, $V = 0$, $W = 0$, is meant the curve

$$J(U, V, W) = \frac{d(U, V, W)}{d(x, y, z)} = \begin{vmatrix} d_xU, d_yU, d_zU \\ d_xV, d_yV, d_zV \\ d_xW, d_yW, d_zW \end{vmatrix} = 0$$

viz., the curve obtained by equating to zero the Jacobian or functional determinant of the functions U, V, W. Some properties of the Jacobian, which are material as to what follows, are mentioned in the Annex No. I.

For the complete statement of the theorem of the variable zomal, it would be necessary to interpret geometrically the condition $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, thereby showing how the single series of the variable zomal is selected out of the double series of the curves $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$ in involution with the given curves. Such a geometrical interpretation of the condition may be sought for as follows, but it is only in a particular case, as afterwards mentioned, that a convenient geometrical interpretation is thereby obtained.

54. Consider the fixed line $\Omega = px + qy + rz = 0$, and let it be proposed to find the locus of the $(r-1)^2$ poles of the line $\Omega = 0$ in regard to the series of curves $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$, where $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$. Take (x, y, z) as the co-ordinates of any one of the poles in question, then in order that (x, y, z) may belong to one of the $(r-1)^2$ poles of the line $\Omega = px + qy + rz = 0$ in regard to the curve $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$, we must have

$$d_x(\mathrm{a}U + \mathrm{b}V + \mathrm{c}W) : d_y(\mathrm{a}U + \mathrm{b}V + \mathrm{c}W) : d_z(\mathrm{a}U + \mathrm{b}V + \mathrm{c}W) = p : q : r :$$

or, what is the same thing—

$$= d_x\Omega : d_y\Omega : d_z\Omega :$$

and these equations give without difficulty

$$\mathrm{a} : \mathrm{b} : \mathrm{c} = J(V, W, \Omega) : J(W, U, \Omega) : J(U, V, \Omega) ,$$

whence, substituting in the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, we have

$$\frac{l}{J(V, W, \Omega)} + \frac{m}{J(W, U, \Omega)} + \frac{n}{J(U, V, \Omega)} = 0$$

as the locus of the $(r-1)^2$ poles in question. Each of the Jacobians is a function of the order $2r - 2$, and the order of the locus is thus $= 4r - 4$. As the given curves $U = 0$, $V = 0$, $W = 0$ belong to the single series of curves, it is clear that the locus passes through the $3(r-1)^2$ points which are the $(r-1)^2$ poles of the fixed line in regard to the curves $U = 0$, $V = 0$, $W = 0$ respectively.

55. In the case where the given trizomal is

$$\sqrt{l(\Theta + L\Phi)} + \sqrt{m(\Theta + M\Phi)} + \sqrt{n(\Theta + N\Phi} = 0 ,$$

$s = r - 1$, that is, where the zomals $\Theta + L\Phi = 0$, $\Theta + M\Phi = 0$, $\Theta + N\Phi = 0$ are each of them curves of the order r, passing through the r intersections of the line $\Phi = 0$ with the curve $\Theta = 0$, then, taking this line $\Phi = 0$ for the fixed line $\Omega = 0$, we have

$$J(V, W, \Omega) = J(\Theta + M\Phi, \Theta + N\Phi, \Phi) = \Phi\{M, N\} ,$$

if, for shortness, $\{M, N\} = J(M - N, \Theta, \Phi) + \Phi J(M, N, \Phi)$, and the like as to the other two Jacobians, so that, attaching the analogous significations to $\{N, L\}$ and $\{L, M\}$, the equation of the locus is

$$\frac{l}{\{M, N\}} + \frac{m}{\{N, L\}} + \frac{n}{\{L, M\}} = 0 ,$$

where observe that each of the curves $\{M, N\} = 0$, $\{N, L\} = 0$, $\{L, M\} = 0$ is a curve of the order $2r-3$; the order of the locus is thus $= 4r-6$, and (as before) this locus passes through the $3(r-1)^2$ points which are the $(r-1)^2$ poles of the line $\Phi = 0$ in regard to the curves $\Theta + L\Phi = 0$, $\Theta + M\Phi = 0$, $\Theta + N\Phi = 0$ respectively.

56. In the case $r = 2$, the trizomal is

$$\sqrt{l(\Theta + L\Phi)} + \sqrt{m(\Theta + M\Phi)} + \sqrt{n(\Theta + N\Phi)} = 0\,,$$

where the zomals are the conics $\Theta + L\Phi = 0$, $\Theta + M\Phi = 0$, $\Theta + N\Phi = 0$, each passing through the same two points $\Theta = 0$, $\Phi = 0$; the locus of the pole of the line $\Phi = 0$, in regard to the variable zomal, is the conic

$$\frac{l}{\{M, N\}} + \frac{m}{\{N, L\}} + \frac{n}{\{L, M\}} = 0\,,$$

viz., $\{M, N\} = 0$, $\{N, L\} = 0$, $\{L, M\} = 0$, are here the lines passing through the poles of the line $\Phi = 0$ in regard to the second and third, the third and first, and the first and second of the given conics respectively: treating l, m, n as arbitrary, the locus is clearly *any* conic through the poles of the line $\Phi = 0$ in regard to the three conics respectively. The Jacobian of the three given conics is a conic related in a special manner to the three given conics, and which might be called the Jacobian conic thereof, and it would be easy to give a complete enunciation of the theorem for the case in hand. (See as to this, Annex No. I, above referred to.) But if, in accordance with the plan adopted in the remainder of the memoir, we at once assume that the points $\Theta = 0$, $\Phi = 0$ are the circular points at infinity, then the theorem can be enunciated under a more simple form—viz., if $\mathbf{A}^0 = 0$, $\mathbf{B}^0 = 0$, $\mathbf{C}^0 = 0$ are the equations of any three circles, then in the trizomal

$$\sqrt{l\mathbf{A}^0} + \sqrt{m\mathbf{B}^0} + \sqrt{n\mathbf{C}^0} = 0\,,$$

the variable zomal is any circle whatever of the series of circles cutting at right angles the orthotomic circle of the three given circles, and having its centre on a certain conic which passes through the centres of the given circles. Moreover, if the co-efficients l, m, n are not given in the first instance, but are regarded as arbitrary, then the last-mentioned conic is any conic whatever through the three centres, and there belongs to such conic and the series of zomals derived therefrom as above, a trizomal curve $\sqrt{l\mathbf{A}^0} + \sqrt{m\mathbf{B}^0} + \sqrt{n\mathbf{C}^0} = 0$. This is obviously the theorem, that if a variable circle has its centre on a given conic, and cuts at right angles a given circle, then the envelope of the variable circle is a trizomal curve $\sqrt{l\mathbf{A}^0} + \sqrt{m\mathbf{B}^0} + \sqrt{n\mathbf{C}^0}$, where $\mathbf{A}^0 = 0$, $\mathbf{B}^0 = 0$, $\mathbf{C}^0 = 0$ are any three circles, positions of the variable circle, and l, m, n are constant quantities depending on the selected three circles.

PART II. (Nos. 57 to 104).—SUBSIDIARY INVESTIGATIONS.

Preliminary Remarks—Art. Nos. 57 and 58.

57. We have just been led to consider the conics which pass through two given points. There is no real loss of generality in taking these to be the circular points at infinity, or say the points I, J—viz., every theorem which in anywise explicitly or implicitly relates to these two points, may, without the necessity of any change in the statement thereof, be understood as a theorem relating instead to any two points P, Q. I call to mind that a circle is a conic passing through the two points I, J, and that lines at right angles to each other are lines harmonically related to the pair of lines from their intersection to the points I, J respectively, so that when (I, J) are replaced by any two given points whatever, the expression a circle must be understood to mean a conic passing through the two given points; and in speaking of lines at right angles to each other, it must be understood that we mean lines harmonically related to the pair of lines from their intersection to the two given points respectively. For instance, the theorem that the Jacobian of any three circles is their orthotomic circle, will mean that the Jacobian of any three conics which each of them passes through the two given points is the orthotomic conic through the same two points, that is, the conic such that at each of its intersections with any one of the three conics, the two tangents are harmonically related to the pair of lines from this intersection to the two given points respectively. Such extended interpretation of any theorem is applicable even to the theorems which involve distances or angles—viz., the terms "distance" and "angle" have a determinate signification when interpreted in reference (not to the circular points at infinity, but instead thereof) to any two given points whatever (see as to this my "Sixth Memoir on Quantics," Nos. 220, *et seq.**) And this being so, the theorem can, without change in the statement thereof, be understood as referring to the two given points.

58. I say then that any theorem (referring explicitly or implicitly) to the circular points at infinity I, J, may be understood as a theorem referring instead to any two given points. We might of course give the theorems in the first instance in terms explicitly referring to the two given points—(viz., instead of a circle, speak of a conic through the two given points, and so in other instances); but, as just explained, this is not really more general, and the theorems would be given in a less concise and familiar form. It would not, on the face of the investigations, be apparent that in treating of the polyzomal curves

$$\sqrt{l(\Theta + L\Phi)} + \sqrt{m(\Theta + M\Phi)} + \&c. = 0,$$

($\Theta = 0$ a conic, $\Phi = 0$ a line, as above), that we were really treating of the curves the zomals whereof are circles, and therein of the theories of foci and focofoci as about to be explained. And for these reasons I shall consider the two points $\Theta = 0$, $\Phi = 0$, to be the circular points at infinity I, J, and in the investigations, &c., make use of the terms circle, right angles, &c., which, in their ordinary significations, have implicit reference to these two points.

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* Phil. Transactions, vol. cxlix. (1859), pp. 61–90. See p. 86.

The present Part does not explicitly relate to the theory of polyzomal curves, but contains a series of researches, partly analytical and partly geometrical, which will be made use of in the following Parts III. and IV. of the Memoir.

The Circular Points at Infinity; Rectangular and Circular Co-ordinates—Art. Nos 59 to 62.

59. The co-ordinates made use of (except in the cases where the general trilinear co-ordinates (x, y, z), or any other co-ordinates, are explicitly referred to), will be either the ordinary rectangular co-ordinates x, y, or else, as we may term them, the circular co-ordinates ξ, η $(=x + iy,\ x - iy$ respectively, $i = \sqrt{-1}$ as usual), but in either case I shall introduce for homogeneity the co-ordinate z, it being understood that this co-ordinate is in fact $= 1$, and that it may be retained or replaced by this its value, in different investigations or stages of the same investigation, as may for the time being be most convenient. In more concise terms, we may say that the co-ordinates are either the rectangular co-ordinates x, y, and $z\ (= 1)$, or else the circular co-ordinates ξ, η, and $z\ (= 1)$. The equation of the line infinity is $z = 0$; the points I, J are given by the equations $(x + iy = 0,\ z = 0)$ and $(x - iy = 0,\ z = 0)$, or, what is the same thing, by the equations $(\xi = 0,\ z = 0)$ and $(\eta = 0,\ z = 0)$ respectively; or in the rectangular co-ordinates the co-ordinates of these points are $(-i, 1, 0)$ and $(i, 1, 0)$ respectively, and in the circular co-ordinates they are $(1, 0, 0)$ and $(0, 1, 0)$ respectively. It is of course, only for points at infinity that the co-ordinate z is $= 0$ (and observe that for any such point the x and y or ξ and η co-ordinates may be regarded as finite); for every point whatever not at infinity the co-ordinate z is, as stated above, $= 1$.

60. Consider a point A, whose co-ordinates (rectangular) are $(a, a', 1)$ and (circular) $(\alpha, \alpha', 1)$, viz., $\alpha = a + a'i$, $\alpha' = a - a'i$; then the equations of the lines through A to the points I, J, are

$$x - az + i(y - a'z) = 0,\quad x - az - i(y - a'z) = 0$$

respectively, or they are

$$\xi - \alpha z = 0 \qquad , \quad \eta - \alpha' z = 0$$

respectively. These equations, if (a, a') or (α, α') are arbitrary, will, it is clear, be the equations of any two lines through the points I, J, respectively.

61. We have from either of the equations in (x, y, z)

$$(x - az)^2 + (y - a'z)^2 = 0,$$

that is, the distance from each other of any two points $(x, y, 1)$, and $(a, a', 1)$ in a line through I or J is $= 0$. And in particular, if $z = 0$, then $x^2 + y^2 = 0$; that is, the distance of the point $(a, a', 1)$ from I or J is in each case $= 0$.

62. Consider for a moment any three points P, Q, A; the perpendicular distance of P from QA is $= 2$ triangle $PQA \div$ distance QA; if Q be any point on the line through A to either of the points I, J, and in particular if Q be either of the points I, J, then the triangle PQA is finite, but the distance QA is $= 0$: that is, the perpendicular distance of P from the line through A to either of the points I, J, that is, from any line through either of these points, is $= \infty$. But, as just stated, the triangle PQA is finite, or say the triangles PIA, PJA are each finite; viz., the co-ordinates (rectangular) of P, A being $(x, y, z = 1)$, $(a, a', 1)$ or (circular) $(\xi, \eta, z = 1)$, $(\alpha, \alpha', 1)$, the expressions for the doubles of these triangles respectively are

$$\begin{vmatrix} x, y, z \\ -i, 1, 0 \\ a, a' 1 \end{vmatrix} \quad , \quad \begin{vmatrix} x, y, z \\ i, 1, 0 \\ a, a', 1 \end{vmatrix}$$

that is, they are (rectangular co-ordinates) $x - az + i(y - a'z)$, $x - az - i(y - a'z)$, or (circular co-ordinates) $\xi - \alpha z$, $\eta - \alpha' z$.

Representing the double areas by PIA, PJA, respectively, and the squared distance of the points A, P by A, we have—

$$\begin{aligned} \mathsf{A} &= (x - az)^2 + (y - a'z)^2 \\ &= (\xi - \alpha z)(\eta - \alpha' z), = PIA, PJA. \end{aligned}$$

Antipoints; Definition and Fundamental Properties—Art. No. 63.

63. Two pairs of points (A, B) and (A_1, B_1) which are such that the lines AB, A_1B_1 bisect each other at right angles in a point O in such wise that $OA = OB = i\,OA_1 = i\,OB_1$, are said to be antipoints, each of the other. In rectangular co-ordinates, taking the co-ordinates of $(A\,B,)$ to be $(a, 0, 1)$ and $(-a, 0, 1)$, those of (A_1, B_1) will be $(0, ai, 1)$ and $(0, -ai, 1)$ respectively, whence joining the points (A, B) with the points (I, J), the points A_1, B_1 are given as the intersections of the lines AI and BJ, and of the lines AJ and BI respectively. Or, what is the same thing, in any quadrilateral wherein I, J are opposite angles, the remaining pairs (A, B) and (A_1, B_1) are antipoints each of the other.

64. In circular co-ordinates, if the co-ordinates of A are $(\alpha, \alpha', 1)$, and those of B are $(\beta, \beta', 1)$, then the equations of

$$\begin{array}{lll} AI, AJ & \text{are} & \xi - \alpha z = 0, \quad \eta - \alpha' z = 0 \\ BI, BJ & \text{,,} & \xi - \beta z = 0, \quad \eta - \beta' z = 0 \end{array}$$

whence the equations of

$$\begin{array}{lll} A_1 I, A_1 J & \text{are} & \xi - \alpha z = 0, \quad \eta - \beta' z = 0 \\ B_1 I, B_1 J & \text{,,} & \xi - \beta z = 0, \quad \eta - \alpha' z = 0\,. \end{array}$$

65. Considering any point P the co-ordinates of which are $\xi, \eta, z\ (=1)$, let $\mathsf{A}, \mathsf{B}, \mathsf{A}_1, \mathsf{B}_1$ be its squared distances from the points A, B, A_1, B_1 respectively; then by what precedes—

$$\begin{aligned}\mathsf{A}\ &= (\xi - \alpha z)\,(\eta - \alpha' z),\\ \mathsf{B}\ &= (\xi - \beta z)\,(\eta - \beta' z),\\ \mathsf{A}_1 &= (\xi - \alpha z)\,(\eta - \beta' z),\\ \mathsf{B}_1 &= (\xi - \beta z)\,(\eta - \alpha' z),\end{aligned}$$

and thence

$$\mathsf{A}\,.\,\mathsf{B} = \mathsf{A}_1\,.\,\mathsf{B}_1\ ;$$

that is, the product of the squared distances of a point P from any two points A, B, is equal to the product of the squared distances of the same point P from the two antipoints A_1, B_1. This theorem, which was, I believe, first given by me in the *Educational Times* (see reprint, vol. vi. 1866, p. 81), is an important one in the theory of foci. It is to be further noticed that we have

$$\begin{aligned}\mathsf{A} + \mathsf{B} - \mathsf{A}_1 - \mathsf{B}_1 &= (\alpha - \beta)\,(\alpha' - \beta')\,z^2\\ &= \qquad\qquad K z^2, = K\,,\end{aligned}$$

if $K, = (\alpha - \alpha')\,(\beta - \beta')$, be the squared distance of the points $A, B, = -$ squared distance of points A_1, B_1.

Antipoints of a Circle—Art. No. 66.

66. A similar notion to that of two pairs of antipoints is as follows, viz., if from the centre of a circle perpendicular to its plane and in opposite senses, we measure off two distances each $= i$ into the radius, the extremities of these distances are antipoints of the circle. It is clear that the antipoints of the circle and the extremities of any diameter thereof are (in the plane of these four points) pairs of antipoints. It is to be added that each antipoint is the centre of a sphere radius zero, or say of a cone sphere, passing through the circle: the circle is thus the intersection of the two cone spheres having their centres at the two antipoints respectively.

Antipoints in relation to a Pair of Orthotomic Circles—Art. No. 67.

67. It is a well-known property that if any circle pass through the points (A, B), and any other circle through the antipoints (A_1, B_1), then these two circles cut at right angles. Conversely if a circle pass through the points A, B, then all the orthotomic circles which have their centres on the line AB pass through the antipoints A_1, B_1. In particular, if on AB as diameter we describe a circle and on $A_1 B_1$ as diameter a circle, then these two circles—being, it is clear, concentric circles with their radii in the ratio $1 : i$, and as concentric circles touching each other at the points (I, J)—cut each other at right angles; or say they are concentric orthotomic circles.

Forms of the Equation of a Circle—Art. Nos. 68 to 71.

68. In rectangular co-ordinates the equation of a circle, co-ordinates of centre $(a, a', 1)$ and radius $= a''$, is

$$\mathsf{A}^\circ = (x-az)^2 + (y-a'z)^2 - a''^2 z^2 = 0\,;$$

and in circular co-ordinates, the co-ordinates of the centre being $(\alpha, \alpha', 1)$, and radius $=a''$ as before, the equation is

$$\mathsf{A}^\circ = (\xi - \alpha z)(\eta - \alpha' z) - a''^2 z^2 = 0.$$

69. I observe in passing, that the origin being at the centre and the radius being $=1$, then writing also $z=1$, the equation of the circle is $\xi\eta=1$, that is the circular co-ordinates of any point of the circle, expressed by means of a variable parameter θ, are $\left(\theta, \frac{1}{\theta}, 1\right)$.

70. Consider a current point P, the co-ordinates of which (rectangular) are $x, y, z\ (=1)$, and (circular) are $\xi, \eta, z\ (=1)$, then the foregoing expression

$$\begin{aligned}\mathsf{A}^\circ &= (x-az)^2 + (y-a'z)^2 - a''^2 z^2 \\ &= (\xi - \alpha z)(\eta - \alpha' z) - a''^2 z^2\end{aligned}$$

denotes, it is clear, the square of the tangential distance of the point P from the circle $\mathsf{A}^\circ = 0$.

71. But there is another interpretation of this same function A°, viz., writing therein $z = 1$, and then

$$\mathsf{A}^\circ = (x-a)^2 + (y-a')^2 + (a''i)^2,$$

we see that A° is the squared distance of P from either of the anti-points of the circle (points lying, it will be recollected, out of the plane of the circle), and we have thus the theorem that the square of the tangential distance of any point P from the circle is equal to the square of its distance from either anti-point of the circle.

On a System of Sixteen Points—Art. Nos. 72 to 77.

72. Take (A, B, C, D) any four concyclic points, and let the anti-points of

$$\begin{array}{lll}(B, C), (A, D) & \text{be} & (B_1, C_1), (A_1, D_1)\,, \\ (C, A), (B, D) & ,, & (C_2, A_2), (B_2, D_2)\,, \\ (A, B), (C, D) & ,, & (A_3, B_3), (C_3, D_3)\,,\end{array}$$

then each of the three new sets (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3) will be a set of four concyclic points.

73. Let O be the centre of the circle through (A, B, C, D), say of the circle O, and then, the lines BC, AD meeting in R, the lines CA, BD in S, and the lines AB, CD in T, let each of these points be made the centre of a circle orthotomic to O, viz., let these new circles be called the circles R, S, T respectively.

As regards the circle R, since its centre lies in BC, the circle passes through (B_1, C_1); and since the centre lies in AD, the circle passes through (A_1, D_1), that is, the four points (A_1, B_1, C_1, D_1) lie in the circle R. Similarly (A_2, B_2, C_2, D_2) lie in the circle S, and $(A_3, B_3, \mathrm{C}_3, D_3)$ in the circle T.

74. The points R, S, T are conjugate points in relation to the circle O; that is, ST, TR, RS are the polars of R, S, T respectively in regard to this circle; and they are, consequently, at right angles to the lines OR, OS, OT respectively; viz., the four centres O, R, S, T are such that the line joining any two of them cuts at right angles the line joining the other two of them, and we see that the relation between the four sets is in fact a symmetrical one; this is most easily seen by consideration of the circular points at infinity I, J, the four sets of points may be arranged thus:—

$$\begin{array}{llll} A\ , & A_3, & A_2, & A_1, \\ B_3, & B\ , & B_1, & B_2, \\ C_2, & C_1, & C\ , & C_3, \\ D_1, & D_2, & D_3, & D\ , \end{array}$$

in such wise that any four of them in the same vertical line pass through I, and any four in the same horizontal line pass through J; and this being so, starting for instance with (A_3, B_3, C_3, D_3) we have anti-points

$$\begin{array}{llll} \text{of } (B_3, C_3), (A_3, D_3) & \text{are} & (B_2, C_2), (A_2, D_2), \\ \text{,,}\ (C_3, A_3), (B_3, D_3) & \text{,,} & (C_1, A_1), (B_1, D_1), \\ \text{,,}\ (A_3, B_3), (C_3, D_2) & \text{,,} & (A\ ,\ B,), (C\ ,\ D\ ), \end{array}$$

and similarly if we start from (A_1, B_1, C_1, D_1) or (A_2, B_2, C_2, D_2).

75. I return for a moment to the construction of (A_1, B_1, C_1, D_1); these are points on the circle R, and (B_1, C_1) are the anti-points of (B, C); that is, they are the intersections of the circle R by the line at right angles to BC from its middle point, or, what is the same thing, by the perpendicular on BC from O. Similarly (A_1, D_1) are the anti-points of (A, D); that is, they are the intersections of the circle R by the perpendicular on AD from O. And the like as to (A_2, B_2, C_2, D_2) and (A_3, B_3, C_3, D_3) respectively.

76. Hence, starting with the points A, B, C, D on the circle O, and constructing as above the circles P, Q, R, and constructing also the perpendiculars from O on the six chords AB, AC, &c.,

$$\begin{array}{llllll} \text{the perpendiculars on} & BC, AD & \text{meet} & \text{circle} & R & \text{in } (B_1, C_1), (A_1, D_1)\,, \\ \text{,,} & CA, BD & \text{,,} & \text{,,} & S & \text{,,}\ (C_2, A_2), (B_2, D_2)\,, \\ \text{,,} & AB, CD & \text{,,} & \text{,,} & T & \text{,,}\ (A_3, B_3), (C_3, D_3)\,, \end{array}$$

so that the whole system is given by means of the circles P, Q, R, and the six perpendiculars.

77. If to fix the ideas (A, B, C, D) are real points taken in order on the real circle O, then the points R, S, T are each of them real; but R and T lie outside, S inside the circle O. The circles R and T are consequently real, but the circle S imaginary, viz., its radius is $= i$ into a real quantity; the imaginary points (A_1, B_1, C_1, D_1) are thus given as the intersections of a real circle by a pair of real lines, and the like as to the imaginary points (A_3, B_3, C_3, D_3); but the

imaginary points (A_2, B_2, C_2, D_2) are only given as the intersections of an imaginary circle (centre real and radius a pure imaginary) by a pair of real lines. The points (C_2, A_2) *quâ* anti-points of (C, A) are easily constructed as the intersections of a real circle by a real line, and the like as to the points (B_2, D_2) *quâ* anti-points of (B, D), but the construction for the two pairs of points cannot be effected by means of the same real circle.

Property in regard to Four Confocal Conics—Art. Nos. 78 to 80.

78. All the conics which pass through the four concyclic points A, B, C, D, have their axes in fixed directions; but three such conics are the line-pairs (BC, AD), (CA, BD), and (AB, CD), whence the directions of the axes are those of the bisectors of the angles formed by any one of these pairs of lines; hence, in particular, considering either axis of a conic through the four points, the lines AB and CD are equally inclined on opposite sides to this axis, and this leads to th theorem that the anti-points (A_3, B_3) (C_3, D_3) are in a conic confocal to the given conic through (A, B, C, D); whence, also, considering any given conic whatever through (A, B, C, D), the points (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2) (A_3, B_3, C_3, D_3) lie severally in three conics, each of them confocal with the given conic.

79. To prove this, consider any two confocal conics, say an ellipse and a hyperbola, and let F be one of their four intersections; join F with the common centre O, and let OT, ON be parallel to the tangent and normal respectively of the ellipse at the point F. OF, OT are in direction conjugate axes of the ellipse, and OF, ON are in direction conjugate axes of the hyperbola; and if they are also the axes in magnitude, that is, if the points T, N are the intersections of OT with the ellipse and of ON with the hyperbola respectively, then it is easy to show that $\overline{OT}^2 + \overline{ON}^2 = 0$. And this being so, imagine on the ellipse any two points A, B such that the chord AB is parallel to OT, that is conjugate to OF; AB is bisected by OF, say in a point K, or we have parallel to OT the semichords or ordinates $KA = KB$; and we may, perpendicularly to this or parallel to ON, draw through K in the hyperbola a chord A_3B_3, which chord will be bisected in K, or we shall have $KA_3 = KB_3$. Hence KA, KA_3 are in the ellipse and the hyperbola respectively ordinates conjugate to the same diameter OF, and the semi-diameters conjugate to OF being OT, ON respectively, we have $KA^2 (= KB^2) : \overline{KA}_3^2 (= \overline{KB}_3^2) = \overline{OT}^2 : \overline{ON}^2$, that is, $\overline{KA}^2 = \overline{KB}^2 = -\overline{KA}_3^2 = -\overline{KB}_3^2$; or (A_3, B_3) will be the anti-points of (A, B).

80. Conversely, if in the ellipse we have the two points (A, B), then drawing the diameter OF conjugate to AB, and through its extremity F, the confocal hyperbola, then the anti-points (A_3, B_3) will lie on the hyperbola. And similarly, if on the ellipse we have the two points (C, D), then drawing the diameter OG conjugate to CD, and through its extremity G a confocal hyperbola, the

anti-points (C_3, D_3) will lie on the hyperbola. Suppose (A, B, C, D) are concyclic, then, as noticed, AB and CD will be equally inclined on opposite sides to the transverse axis of the ellipse—the conjugate diameters OF, OG will therefore be equally inclined on opposite sides of the transverse axis—and the points F and G will therefore be situate symmetrically on opposite sides of the transverse axis, that is, the points F and G will respectively determine the same confocal hyperbola, and we have thus the required theorem, viz., if (A, B, C, D) are any four concyclic points on an ellipse, or say on a conic, and if (A_3, B_3) are the anti-points of (A, B), and (C_3, D_3) the anti-points of (C, D), then (A_3, B_3, C_3, D_3) will lie on a conic confocal with the given conic.

System of the Sixteen Points, the Axial Case—Art. Nos. 81 to 85.

81. The theorems hold good when the four points A, B, C, D are in a line; the anti-points (B_1, C_1) of (B, C), &c., are in this case situate symmetrically on opposite sides of the line, so that it is evident at sight that we have (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3), each set in a circle; and that the centres R, S, T of these circles lie in the line. The construction for the general case becomes, however, indeterminate, and must therefore be varied. If in the general case we take any circle through (B, C), and any circle through (A, D), then the circle R cuts at right angles these two circles, and has, consequently, its centre R in the radical axis of the two circles; whence, when the four points are in a line, taking any circle through (B, C), or in particular the circle on BC as diameter, and any circle through (A, D), or in particular the circle on AD as diameter,—the radical axis of these two circles intersects the line in the required centre R, and the circle R is the circle with this centre cutting at right angles the two circles respectively; the circles S and T are, of course, obtained by the like construction in regard to the combinations $(C, A\,;\, B, D)$ and $(A, B\,;\, C, D$, respectively. It may be added, that we have

$$\left.\begin{array}{c}R\\ S\\ T\end{array}\right\}\text{centre and}\left\{\begin{array}{l}\text{extremities } R\\ \text{of diameter } S\\ \text{of circles } \quad T\end{array}\right\}\text{sibiconjugate points of involutions}\left\{\begin{array}{l}B, C;\ A, D,\\ C, A,\ B, D,\\ A, B,\ C, D,\end{array}\right.$$

and that (as in the general case) the circles R, S, T intersect each pair of them at right angles; and they are evidently each intersected at right angles by the line $ABCD$ (or axis of the figure), which replaces the circle O in the general case.

82. If the points A, B, C, D are taken in order on the line, then the points R, S, T are all real, viz., the point R is situate, on one side or the other, outside AD, but the points S and T are each of them situate between B and C; the circles R and T are real, but the circle S has its radius a pure imaginary quantity.

83. If one of the four points, suppose D, is at infinity on the line, then the anti-points of (A, D), of (B, D), and of (C, D) are each of them the two points

(I, J). It would at first sight appear that the only conditions for the circles R, S, T were the conditions of passing through the anti-points of (B, C), of (C, A), and of (A, B) respectively, and that these circles thus became indeterminate; but in fact the definition of the circles is then as follows, viz., R has its centre at A, and passes through the anti-points of (B, C): (whence squared radius $=AB \, . \, AC$). And similarly, S has its centre at B, and passes through anti-points of (C, A), (squared radius $= BA \, . \, BC$); and T has its centre at C, and passes through anti-points of (A, B), (squared radius $= CA \, . \, CB$); these three circles cut each other at right angles. As before, A, B, C being in order on the line, the circles R, T are real, but the circle S has its radius a pure imaginary quantity.

84. That the circles are as just mentioned appears as follows: taking the line as axis of x, and a, b, c, d for the x co-ordinates of the four points respectively, then the co-ordinates of A_1, D_1 are

$$\tfrac{1}{2}(a + d), \pm \tfrac{1}{2}i(a - d);$$

whence, m being arbitrary, the general equation of a circle through A_1, D_1 is

$$x^2 + y^2 - 2mxz + [m(a + d) - ad]z^2 = 0 \, ,$$

writing herein $$m = a - \frac{k^2}{d}$$ this becomes

$$x^2 + y^2 - 2\left(a - \frac{k^2}{d}\right)xz + \left(a^2 - k^2 - \frac{ak^2}{d}\right)z^2 = 0 \, ,$$

viz., for $d = \infty$ it is

$$(x - az)^2 + y^2 - k^2z^2 = 0 \, ,$$

which is a circle having A for its centre, and its radius an arbitrary quantity k. If the circle passes through the anti-points of B, C, the co-ordinates of these are

$$\tfrac{1}{2}(b + c), \pm \tfrac{1}{2}i(b - c) \, ,$$

and we find

$$k^2 = [\tfrac{1}{2}(b + c) - a]^2 - \tfrac{1}{4}(b - c)^2 = (a - b)(a - c) \, .$$

85. Reverting to the general case of four points A, B, C, D on a line, the theorem as to the confocal conics holds good under the form that, drawing any conic whatever through (A_1, B_1, C_1, D_1) the points (A_2, B_2, C_2, D_2), and (A_3, B_3, C_3, D_3) lie in confocal conics, these conics have their centre on the line, and axes in the direction of and perpendicular to the line. When D is at infinity, the confocal conics become any three concentric circles through (B_1, C_1), (C_2, A_2), and (A_3, B_3) respectively.

The Involution of Four Circles.—Art. Nos. 86 to 91.

86. Consider any four points A, B, C, D, the centres of circles denoted by these same letters, and let **A°, B°, C°, D°** signify as usual, viz., if (in orthogonal

co-ordinates) $(a, a', 1)$ are the co-ordinates of the centre, and a'' the radius of the circle A, then $\mathbf{A}^{\circ}$ stands for $(x-az)^2+(y-a'z)^2-a''^2z^2$, and the like for $\mathbf{B}^{\circ}$, $\mathbf{C}^{\circ}$, $\mathbf{D}^{\circ}$. Write also

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d} = BCD : -\,CDA : DAB : -\,ABC\,,$$

where BCD, &c., are the triangles formed by the points (B, C, D), &c.; the analytical expressions are

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d} = \begin{vmatrix} b, & b', & 1 \\ c, & c', & 1 \\ d, & d', & 1 \end{vmatrix} : -\begin{vmatrix} c, & c', & 1 \\ d, & d', & 1 \\ a, & a', & 1 \end{vmatrix} : \begin{vmatrix} d, & d', & 1 \\ a, & a', & 1 \\ b, & b', & 1 \end{vmatrix} : -\begin{vmatrix} a, & a', & 1 \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix}$$

so that

$$\begin{aligned} \mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} &= 0\,, \\ \mathrm{a}a + \mathrm{b}b + \mathrm{c}c + \mathrm{d}d &= 0\,, \\ \mathrm{a}a' + \mathrm{b}b' + \mathrm{c}c' + \mathrm{d}d' &= 0\,; \end{aligned}$$

this being so, it is clear that we have

$$\mathrm{a}\mathbf{A}^{\circ} + \mathrm{b}\mathbf{B}^{\circ} + \mathrm{c}\mathbf{C}^{\circ} + \mathrm{d}\mathbf{D}^{\circ} =$$
$$z^2[\mathrm{a}(a^2+a'^2-a''^2) + \mathrm{b}(b^2+b'^2-b''^2) + \mathrm{c}(c^2+c'^2-c''^2) + \mathrm{d}(d^2+d'^2-d''^2)] = Kz^2, = K\,,$$

a constant.

87. I am not aware that in the general case there is any convenient expression for this constant K; it is $= 0$ when the four circles have the same orthotomic circle; in fact, taking as origin the centre of the orthotomic circle, and its radius to be $= 1$, we have

$$a^2 + a'^2 - a''^2 = 1, \text{ \&c.}\,,$$

whence

$$K = \mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} = 0\,;$$

that is, if the circles A, B, C, D have the same orthotomic circle, then $\mathbf{A}^{\circ}$, $\mathbf{B}^{\circ}$, $\mathbf{C}^{\circ}$, $\mathbf{D}^{\circ}$, a, b, c, d, signifying as above, we have

$$\mathrm{a}\mathbf{A}^{\circ} + \mathrm{b}\mathbf{B}^{\circ} + \mathrm{c}\mathbf{C}^{\circ} + \mathrm{d}\mathbf{D}^{\circ} = 0\,,$$

and, in particular, if the circles reduce themselves to the points A, B, C, D respectively, then (writing as usual $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ in place of $\mathbf{A}^{\circ}$, $\mathbf{B}^{\circ}$, $\mathbf{C}^{\circ}$, $\mathbf{D}^{\circ}$) if the four points A, B, C, D are on a circle, we have

$$\mathrm{a}\mathbf{A} + \mathrm{b}\mathbf{B} + \mathrm{c}\mathbf{C} + \mathrm{d}\mathbf{D} = 0\,.$$

88. This last theorem may be regarded as a particular case of the theorem

$$\mathrm{a}\mathbf{A} + \mathrm{b}\mathbf{B} + \mathrm{c}\mathbf{C} + \mathrm{d}\mathbf{D} = Kz^2 = K,$$

viz., the four circles reducing themselves to the points A, B, C, D, we can find for the constant K an expression which will of course vanish when the points are on a circle. For this purpose, let the lines BC, AD meet in R, the lines

CA, *BD* in *S*, and the lines *AB*, *CD* in *T*; we may, to fix the ideas, consider *ABCD* as forming a convex quadrilateral, *R* and *T* will then be the exterior centres, *S* the interior centre; a, b, c, d, may be taken *equal* to *BCD*, − *CDA*, *DAB*, − *ABC*, where the areas *BCD*, &c., are each taken positively. The expression $\mathrm{a}\mathbf{A} + \mathrm{b}\mathbf{B} + \mathrm{c}\mathbf{C} + \mathrm{d}\mathbf{D}$ has the same value, whatever is the position of the point $P\,(x, y, z = 1)$; taking this point at *R*, and writing for a moment

$$RA = \alpha, RB = \beta, RC = \gamma, RD = \delta,$$

then

$$BCD = (RCD - RBD) = \tfrac{1}{2}RD\,(RC - RB)\sin R = (\gamma-\beta)\delta\sin R\,,$$

with similar expressions for the other triangles; and we thus have

$$\mathrm{a}\mathbf{A} + \mathrm{b}\mathbf{B} + \mathrm{c}\mathbf{C} + \mathrm{d}\mathbf{D} = \tfrac{1}{2}z^2\,.\sin R\left\{\begin{array}{l} \alpha^2(\gamma-\beta)\delta \\ -\beta^2(\delta-\alpha)\gamma \\ +\gamma^2(\delta-\alpha)\beta \\ -\delta^3(\gamma-\beta)\alpha \end{array}\right\} = \tfrac{1}{2}z^2\sin R\,(\beta\gamma - \alpha\delta)(\gamma-\beta)(\delta-\alpha)\,,$$

that is, replacing $\alpha, \beta, \gamma, \delta$, by their values, and writing also $z = 1$, we have

$$\mathrm{a}\mathbf{A} + \mathrm{b}\mathbf{B} + \mathrm{c}\mathbf{C} + \mathrm{d}\mathbf{D} = \tfrac{1}{2}\sin R\,.\,(RB\,.\,RC - RA\,.\,RD)BC\,.\,AD\,,$$

where $\tfrac{1}{2}\sin R.BC.AD$ is in fact the area of the quadrilateral *ABCD*; we have thus

$$\begin{aligned}\mathrm{a}\mathbf{A} + \mathrm{b}\mathbf{B} + \mathrm{c}\mathbf{C} + \mathrm{d}\mathbf{D} &= (RB.RC - RA.RD)\square \\ &= (SC.SA - SB.SD)\square \\ &= (TA.TB - TC.TD)\square\end{aligned}$$

where it is to be observed that *SA*, *SC* being measured in opposite directions from *S*, must be considered, one as positive, the other as negative, and the like as regards *SB*, *SD*. This expression for the value of the constant is due to Mr Crofton. In the particular case where *A*, *B*, *C*, *D*, are on a circle, we have as before

$$\mathrm{a}\mathbf{A} + \mathrm{b}\mathbf{B} + \mathrm{c}\mathbf{C} + \mathrm{d}\mathbf{D} = 0\,.$$

89. If the four points *A*, *B*, *C*, *D*, are on a circle, then, taking as origin the centre of this circle and its radius as unity, the circular co-ordinates of the four points will be

$$\left(\alpha, \frac{1}{\alpha}, 1\right),\ \left(\beta, \frac{1}{\beta}, 1\right),\ \left(\gamma, \frac{1}{\gamma}, 1\right),\ \left(\delta, \frac{1}{\delta}, 1\right),$$

the corresponding forms of $\mathbf{A}^\circ$, &c., being

$$\mathbf{A}^\circ = (\xi - \alpha z)\left(\eta - \frac{1}{\alpha}z\right) - a''^2z^2, \text{ \&c.}$$

the expressions for a, b, c, d, observing that we have

$$\left|\begin{array}{l} \beta, \beta^{-1}, 1 \\ \gamma, \gamma^{-1}, 1 \\ \delta, \delta^{-1}, 1 \end{array}\right| = \frac{1}{\beta\gamma\delta}\left|\begin{array}{l} 1, \beta, \beta^2 \\ 1, \gamma, \gamma^2 \\ 1, \delta, \delta^2 \end{array}\right| = \frac{1}{\beta\gamma\delta}(\beta\gamma\delta), \text{ \&c.}$$

if $(\beta\gamma\delta)$, &c., denote $(\beta-\gamma)(\gamma-\delta)(\delta-\beta)$, &c., become

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=\alpha(\beta\gamma\delta):-\beta(\gamma\delta\alpha):\gamma(\delta\alpha\beta):-\delta(\alpha\beta\gamma),$$

which are convenient formulæ for the case in question.

90. If the points A, B, C, D, are on a line, then taking this line for the axis of x, we may write $\mathsf{A}^\circ=(x-az)^2+y^2-a''^2z^2$, &c. It is to be remarked here that we can, without any relation whatever between the radii of the circles, satisfy the equation

$$\mathrm{a}\mathsf{A}^\circ+\mathrm{b}\mathsf{B}^\circ+\mathrm{c}\mathsf{C}^\circ+\mathrm{d}\mathsf{D}^\circ=0;$$

in fact this will be the case if we have

$$\begin{array}{llll}\mathrm{a} & +\mathrm{b} & +\mathrm{c} & +\mathrm{d}=0,\\ \mathrm{a}a & +\mathrm{b}b & +\mathrm{c}c & +\mathrm{d}d=0,\\ \mathrm{a}(a^2-a''^2) & +\mathrm{b}(b^2-b''^2) & +\mathrm{c}(c^2-c''^2) & +\mathrm{d}(d^2-d''^2)=0,\end{array}$$

equations which determine the ratios a : b : c : d. In the case where the circles reduce themselves to the points A, B, C, D, these equations become

$$\begin{array}{l}\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0,\\ \mathrm{a}a+\mathrm{b}b+\mathrm{c}c+\mathrm{d}d=0,\\ \mathrm{a}a^2+\mathrm{b}b^2+\mathrm{c}c^2+\mathrm{d}d^2=0,\end{array}$$

giving

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=(bcd):-(cda):(dab):-(abc);$$

if for shortness (bcd), &c. stand for $(b-c)(c-d)(d-b)$, &c.; and for these values, we have

$$\mathrm{a}\mathsf{A}+\mathrm{b}\mathsf{B}+\mathrm{c}\mathsf{C}+\mathrm{d}\mathsf{D}=0.$$

91. A very noticeable case is when the four circles are such that the foregoing values of (a, b, c, d) also satisfy the equation

$$\mathrm{a}\mathsf{A}^\circ+\mathrm{b}\mathsf{B}^\circ+\mathrm{c}\mathsf{C}^\circ+\mathrm{d}\mathsf{D}^\circ=0;$$

the condition for this is obviously

$$\mathrm{a}a''^2+\mathrm{b}b''^2+\mathrm{c}c''^2+\mathrm{d}d''^2=0;$$

or, as it may also be written,

$$\frac{a''^2}{(a-b)(a-c)(a-d)}+\frac{b''^2}{(b-c)(b-d)(b-a)}+\frac{c''^2}{(c-d)(c-a)(c-b)}+\frac{d''^2}{(d-a)(d-b)(d-c)}=0.$$

On a Locus connected with the foregoing Properties.—Art. No. 92.

92. If, as above, A, B, C, D are any four points, and $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ are the squared distances of a current point P from the four points respectively, then the locus of the foci of the conics which pass through the four points is the tetrazomal curve

$$\mathrm{a}\sqrt{\mathsf{A}}+\mathrm{b}\sqrt{\mathsf{B}}+\mathrm{c}\sqrt{\mathsf{C}}+\mathrm{d}\sqrt{\mathsf{D}}=0.$$

In fact the sum $\mathrm{a}\mathsf{A}+\mathrm{b}\mathsf{B}+\mathrm{c}\mathsf{C}+\mathrm{d}\mathsf{D}$ has, it has been seen, a constant value for all positions of the point P; taking P to be the other focus, its squared distances are $(k-\sqrt{A})^2$, &c., whence for the first-mentioned focus we have

$$\mathrm{a}\mathsf{A}+\mathrm{b}\mathsf{B}+\mathrm{c}\mathsf{C}+\mathrm{d}\mathsf{D}=\mathrm{a}(k-\sqrt{\mathsf{A}})^2+\mathrm{b}(k-\sqrt{\mathsf{B}})^2+\mathrm{c}(k-\sqrt{\mathsf{C}})^2+\mathrm{d}(k-\sqrt{\mathsf{D}})^2\,;$$

or recollecting that $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0$, we have for the locus in question $\mathrm{a}\sqrt{\mathsf{A}}+\mathrm{b}\sqrt{\mathsf{B}}+\mathrm{c}\sqrt{\mathsf{C}}+\mathrm{d}\sqrt{\mathsf{D}}=0$; this locus will be discussed in the sequel. I remark here, that in the case where the four points are on a circle, then (as mentioned above), the axes of the several conics are in the same fixed directions; there are thus two sets of foci, those on the axis in one direction, and those on the axis in the other direction; it might therefore be anticipated, and it will appear, that in this case the tetrazomal breaks up into two trizomal curves.

Formulæ as to the two Sets (A, B, C, D), *and* (A_1, B_1, C_1, D_1), *each of four Concyclic Points*—Art. Nos. 93 to 98.

93. Consider the four points A, B, C, D on a circle, then taking, as before, their circular co-ordinates to be $(\alpha, \alpha', 1)$, $(\beta, \beta', 1)$, $(\gamma, \gamma', 1)$, $(\delta, \delta', 1)$, the condition that the points may be on a circle is

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix}=0\,,$$

viz., this equation may be written

$$\begin{aligned}&(\beta-\gamma)(\alpha-\delta):(\gamma-\alpha)(\beta-\delta):(\alpha-\beta)(\gamma-\delta)\\ =\;&(\beta'-\gamma')(\alpha'-\delta'):(\gamma'-\alpha')(\beta'-\delta'):(\alpha'-\beta')(\gamma'-\delta')\,;\end{aligned}$$

or, if for shortness, we take

$$\begin{aligned} a&=\beta-\gamma, & f&=\alpha-\delta, & a'&=\beta'-\gamma', & f'&=\alpha'-\delta'\,,\\ b&=\gamma-\alpha, & g&=\beta-\delta, & b'&=\gamma'-\alpha', & g'&=\beta'-\delta',\\ c&=\alpha-\beta, & h&=\gamma-\delta, & c'&=\alpha'-\beta', & h'&=\gamma'-\delta',\end{aligned}$$

and consequently

$$\begin{aligned} af+bg+ch&=0\,, & a'f'+b'g'+c'h'&=0\,,\\ a&=g-h\,, & a'&=g'-h'\,,\\ b&=h-f\,, & b'&=h'-f'\,,\\ c&=f-g\,, & c'&=f'-g'\,,\\ a+b+c&=0\,, & a'+b'+c'&=0\,,\end{aligned}$$

then the equation is

$$af:bg:ch=a'f':b'g':c'h'\,.$$

94. Let a, b, c, d, denote as before $(\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=BCD:-CDA:DAB:-ABC)$, then we have

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=\begin{vmatrix}\beta, & \beta', & 1\\ \gamma, & \gamma', & 1\\ \delta, & \delta', & 1\end{vmatrix}:-\begin{vmatrix}\gamma, & \gamma', & 1\\ \delta, & \delta', & 1\\ \alpha, & \alpha', & 1\end{vmatrix}:\begin{vmatrix}\delta, & \delta', & 1\\ \alpha, & \alpha', & 1\\ \beta, & \beta', & 1\end{vmatrix}:-\begin{vmatrix}\alpha, & \alpha', & 1\\ \beta, & \beta', & 1\\ \gamma, & \gamma', & 1\end{vmatrix}$$

and we may write

$$\begin{array}{llll}
\mathrm{a} = & \quad . & ah' - a'h, & ag' - a'g, & gh' - g'h\,, \\
\mathrm{b} = bh' - b'h, & \quad . \quad , & bf' - b'f, & hf' - h'f\,, \\
\mathrm{c} = cg' - c'g, & cf' - c'f, & \quad . \quad , & fg' - f'g\,, \\
\mathrm{d} = cb' - c'b, & ac' - a'c, & ba' - b'a, & \quad .
\end{array}$$

viz., the expressions in the same horizontal line are equal, and a, b, c, d are proportional to the expressions in the four lines respectively.

95. I say that we have

$$\frac{c'f'}{ah}\mathrm{a} = \frac{c'g'}{bh}\mathrm{b} = \frac{a'f'}{af}\mathrm{c} = \frac{f'g'}{ab}\mathrm{d}\,,$$

viz., this will be the case if

$$\begin{aligned}
bc'\mathrm{a} &= hg'\mathrm{d}\,,\\
ac'\mathrm{b} &= hf'\mathrm{d}\,,\\
a'b\mathrm{c} &= fg'\mathrm{d}\,,
\end{aligned}$$

and selecting the convenient expressions for a, b, c, d, these equations become

$$\begin{aligned}
bc'\,(gh' - g'h) &= g'h\,(cb' - c'b)\,,\\
ac'\,(hf' - h'f) &= f'h\,(ac' - a'c)\,,\\
a'b\,(fg' - f'g) &= fg'\,(ba' - b'a)\,,
\end{aligned}$$

viz , these equations are respectively $bgc'h' = b'g'ch$, $cha'f' = c'h'af$, $afb'g' = a'f'bg$, and are consequently satisfied. It thus appears that the equation

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0$$

is transformable into

$$\frac{c'f'}{ah}\,l + \frac{c'g'}{bh}\,m + \frac{a'f'}{af}\,n + \frac{f'g'}{ab}\,p = 0\,,$$

which is of course one of a system of similar forms.

96. Take (A_1, D_1) the anti-points of A, D; (B_1, C_1) the anti-points of (B, C); or say that the circular co-ordinates of A_1, B_1, C_1, D_1 are $(\alpha, \delta', 1)$, $(\beta, \gamma', 1)$, $(\gamma, \beta', 1)$, $(\delta, \alpha', 1)$ respectively; the points A_1, B_1, C_1, D_1 are, as above mentioned, on a circle, the condition that this may be so being in fact

$$\begin{vmatrix}
1, & \alpha, & \delta', & \alpha\delta' \\
1, & \beta, & \gamma' & \beta\gamma' \\
1, & \gamma, & \beta' & \gamma\beta' \\
1, & \delta, & \alpha', & \delta\alpha'
\end{vmatrix} = 0,$$

equivalent to

$$af : bg : ch = a'f' : b'g' : c'h'\,.$$

97. Let $(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1, \mathrm{d}_1)$ be the corresponding quantities to (a, b, c, d), viz. $\mathrm{a}_1 : \mathrm{b}_1 : \mathrm{c}_1 : \mathrm{d}_1 = B_1C_1D_1 : -\,C_1D_1A_1 : D_1A_1B_1 : -\,A_1B_1C_1$; we have

$$\mathrm{a}_1 : \mathrm{b}_1 : \mathrm{c}_1 : \mathrm{d}_1 = \begin{vmatrix} \beta, & \gamma', & 1 \\ \gamma, & \beta', & 1 \\ \delta, & \alpha', & 1 \end{vmatrix} : - \begin{vmatrix} \gamma, & \beta', & 1 \\ \delta, & \alpha', & 1 \\ \alpha, & \delta', & 1 \end{vmatrix} : \begin{vmatrix} \delta, & \alpha', & 1 \\ \alpha, & \delta', & 1 \\ \beta, & \gamma', & 1 \end{vmatrix} : - \begin{vmatrix} \alpha, & \delta', & 1 \\ \beta, & \gamma', & 1 \\ \gamma, & \beta', & 1 \end{vmatrix}$$

giving rise to a similar set of forms

$$\begin{array}{ll}
\mathrm{a}_1 = & \quad . \quad , -ac' + ha', \quad a'g + b'a, -c'g - b'h\,, \\
\mathrm{b}_1 = & -c'b - g'h, \quad . \quad , -f'b - g'f, -f'h + c'f\,, \\
\mathrm{c}_1 = & b'c + h'g, -f'c + h'f, \quad . \quad , \quad f'g + g'f\,, \\
\mathrm{d}_1 = & g'c + h'b, -h'a + a'c, -a'b - g'a, \quad . \quad ,
\end{array}$$

and leading to

$$\frac{cf}{a'c'}\,\mathrm{a}_1 = -\frac{cg}{c'g'}\,\mathrm{b}_1 = \frac{af}{a'f'}\,\mathrm{c}_1 = -\frac{fg}{a'g'}\,\mathrm{d}_1\,,$$

so that the equation $\frac{l_1}{\mathrm{a}_1} + \frac{m_1}{\mathrm{b}_1} + \frac{n_1}{\mathrm{c}_1} + \frac{p_1}{\mathrm{d}_1} = 0$, is transformable into

$$\frac{cf}{a'c'}\,l_1 - \frac{cg}{c'g'}\,m_1 + \frac{af}{a'f'}\,n_1 - \frac{fg}{a'g'}\,p_1 = 0\,.$$

98. Let A, B, C, D, be, as above, points on a circle; (A_1, D_1) and (B_1, C_1) the anti-points of (A, B), (B, C) respectively. Write

$$\begin{array}{ll}
\mathsf{A} = (\xi - \alpha z)(\eta - \alpha' z)\,, & \mathsf{A}_1 = (\xi - \alpha z)(\eta - \delta' z)\,, \\
\mathsf{B} = (\xi - \beta z)(\eta - \beta' z)\,, & \mathsf{B}_1 = (\xi - \beta z)(\eta - \gamma' z)\,, \\
\mathsf{C} = (\xi - \gamma z)(\eta - \gamma' z)\,, & \mathsf{C}_1 = (\xi - \gamma z)(\eta - \beta' z)\,, \\
\mathsf{D} = (\xi - \delta z)(\eta - \delta' z)\,, & \mathsf{D}_1 = (\xi - \delta z)(\eta - \delta' z)\,;
\end{array}$$

then we have identically

$$\begin{array}{l}
(\delta - \alpha)(\delta' - \alpha')\,\mathsf{B} = (\beta - \delta)(\beta' - \delta')\,\mathsf{A} + (\beta - \alpha)(\beta' - \alpha')\,\mathsf{D} - (\beta - \delta)(\beta' - \alpha')\,\mathsf{A}_1 - (\beta - \alpha)(\beta' - \delta')\,\mathsf{D}_1\,, \\
(\delta - \alpha)(\delta' - \alpha')\,\mathsf{C} = (\gamma - \delta)(\gamma' - \delta')\,\mathsf{A} + (\gamma - \alpha)(\gamma' - \alpha')\,\mathsf{D} - (\gamma - \delta)(\gamma' - \alpha')\,\mathsf{A}_1 - (\gamma - \alpha)(\gamma' - \delta')\,\mathsf{D}_1\,, \\
(\delta - \alpha)(\delta' - \alpha')\,\mathsf{B}_1 = (\beta - \delta)(\gamma' - \delta')\,\mathsf{A} + (\beta - \alpha)(\gamma' - \alpha')\,\mathsf{D} - (\beta - \delta)(\gamma' - \alpha')\,\mathsf{A}_1 - (\beta - \alpha)(\gamma' - \delta')\,\mathsf{D}_1\,, \\
(\delta - \alpha)(\delta - \alpha')\,\mathsf{C}_1 = (\gamma - \delta)(\beta' - \delta')\,\mathsf{A} + (\gamma - \alpha)(\beta' - \alpha')\,\mathsf{D} = (\gamma - \delta)(\beta' - \alpha')\,\mathsf{A}_1 - (\gamma - \alpha)(\beta' - \delta')\,\mathsf{D}_1\,,
\end{array}$$

or in the foregoing notation

$$\begin{array}{l}
ff'\,\mathsf{B} = gg'\mathsf{A} + cc'\mathsf{D} + gc'\mathsf{A}_1 + cg'\mathsf{D}_1\,, \\
ff'\,\mathsf{C} = hh'\mathsf{A} + bb'\mathsf{D} - hb'\mathsf{A}_1 - bh'\mathsf{D}_1\,, \\
ff'\,\mathsf{B}_1 = gh'\mathsf{A} - cb'\mathsf{D} - gb'\mathsf{A}_1 + ch'\mathsf{D}_1\,, \\
ff'\,\mathsf{C}_1 = g'h\mathsf{A} - bc'\mathsf{D} + hc'\mathsf{A}_1 - bg'\mathsf{D}_1\,.
\end{array}$$

Further Properties in relation to the same Sets (A, B, C, D) *and* (A_1, B_1, C_1, D_1)— Art. Nos. 99 to 104.

99. It is be shown that in virtue of these equations, and if moreover $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0$, then it is possible to find l_1, m_1, n_1, p_1, such that we have identically

$$-l\mathsf{A} + m\mathsf{B} + n\mathsf{C} - p\mathsf{D} + l_1\mathsf{A}_1 - m_1\mathsf{B}_1 - n_1\mathsf{C}_1 + p_1\mathsf{D}_1 = 0\,.$$

This equation will in fact be identically true if only

$$\begin{array}{lllllll}
-ff'l + gg'm + hh'n & & . & - gh'm_1 & - g'h\,n_1 & & = 0\,, \\
\phantom{-ff'l +{}} cc'm + bb'n & - ff'p & . & + cb'm_1 & + bc'\,n_1 & & = 0\,, \\
\phantom{-ff'l +{}} gc'm - hb'n & & + ff'l_1 & + gb'm_1 & - hc'\,n_1 & & = 0\,, \\
\phantom{-ff'l +{}} cg'm - bh'n & & . & + ch'm_1 & + bg'\,n_1 & + ff'p_1 & = 0\,.
\end{array}$$

From the first and second equations eliminating m_1 or n_1, the other of these quantities disappears of itself, and we thus obtain two equations which must be equivalent to a single one, viz., we have

$$bc'ff'l + c'g'afm + bha'f'n + g'hff'p = 0\,,$$
$$b'cff'l + cga'f'm + b'h'afn + gh'ff'p = 0\,;$$

which equations may also be written

$$\frac{c'f'}{ah}l + \frac{c'g'}{bh}m + \frac{a'f'}{af}n + \frac{f'g'}{ab}p = 0\,,$$
$$\frac{cf}{a'h'}l + \frac{cg}{b'h'}m + \frac{af}{a'f'}n + \frac{fg}{a'b'}p = 0\,;$$

and it thus appears that the equations are equivalent to each other, and to the assumed relation

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0\,.$$

100. Similarly, from the third and fourth equations eliminating m or n, the other of these quantities disappears of itself, and we find

$$cg'ff'l_1 - cga'f'm_1 + afc'g'n_1 - c'gff'p_1 = 0\,,$$
$$bh'ff'l_1 - afb'h'm_1 + bha'f'n_1 - b'hff'p_1 = 0\,,$$

equations which may be written

$$\frac{cf}{a'c'}l - \frac{cg}{c'g'}m + \frac{af}{a'f'}n - \frac{fg}{g'a'}p = 0\,,$$
$$\frac{f'h'}{ah}l - \frac{b'h'}{bg}m + \frac{a'f'}{af}n - \frac{b'g'}{ab}p = 0\,,$$

where we see that the two equations are equivalent to each other and to the equation

$$\frac{l_1}{\mathrm{a}_1} + \frac{m_1}{\mathrm{b}_1} + \frac{n_1}{\mathrm{c}_1} + \frac{p_1}{\mathrm{d}_1} = 0\,.$$

It thus appears that the quantities l_1, m_1, n_1, p_1, must satisfy this last equation. It is to be observed that the first and second equations being, as we have seen, equivalent to a single equation, either of the quantities m_1, n_1, may be assumed at pleasure, but the other is then determined; the third and fourth equations then give l_1, p_1; and the quantities l_1, m_1, n_1, p_1, so obtained, satisfy *identically* the equation $\frac{l_1}{\mathrm{a}_1} + \frac{m_1}{\mathrm{b}_1} + \frac{n_1}{\mathrm{c}_1} + \frac{p_1}{\mathrm{d}_1} = 0$.

101. Now writing

$$ff'l_1 = -g(c'm + b'm_1) + h(b'n + c'n_1) ,$$
$$ff'p_1 = -c(g'm - h'm_1) + b(h'n - g'n_1) ,$$

and

$$ff'p = \quad c(c'm + b'm_1) + b(b'n + c'n_1) ,$$
$$ff'l = \quad g(g'm - h'm_1) + h(h'n - g'n_1) ,$$

we find

$$\begin{aligned} f^2f'^2(l_1p_1 - lp) &= -(bg + ch)[(c'm + b'm_1)(h'n - g'n_1) + (g'm - h'm_1)(b'n + c'n_1)] \\ &= (bg + ch)(b'g' + c'h')(m_1n_1 - mn) \\ &= aa'ff'(m_1n_1 - mn) \end{aligned}$$

that is,

$$ff'(l_1p_1 - lp) = aa'(m_1n_1 - mn)$$

viz., this equation is satisfied identically by the values of l_1, m_1, n_1, p_1 determined as above.

102. Hence if $m_1 n_1 = mn$, we have also $l_1 p_1 = lp$, and we can determine m_1, n_1, so that m_1n_1 shall $= mn$, viz., in the first or second of the four equations (these two being equivalent to each other, as already mentioned), writing $m_1 = \theta n$, and therefore $n_1 = \frac{1}{\theta} m$, we have

$$-ff'l + gg'm + hh'n - gh'n\theta - g'hm\frac{1}{\theta} = 0 ,$$

$$cc'm + bb'n - ff'p + cb'n\theta + bc'm\frac{1}{\theta} = 0 ,$$

which are, in fact, the same quadric equation in θ, viz., we have

$$\frac{-ff'l + gg'm + hh'n}{cc'm + bb'n - ff'p} = -\frac{gh'}{cb'} = -\frac{g'h}{bc'} .$$

The final result is that there are two sets of values of l_1, m_1, n_1, p_1, each satisfying the identity

$$-l\mathsf{A} + m\mathsf{B} + n\mathsf{C} - p\mathsf{D} + l_1\mathsf{A}_1 - m_1\mathsf{B}_1 - n_1\mathsf{C}_1 + p_1\mathsf{D}_1 = 0 ,$$

and for each of which we have

$$\frac{l_1}{\mathrm{a}_1} + \frac{m_1}{\mathrm{b}_1} + \frac{n_1}{\mathrm{c}} + \frac{p_1}{\mathrm{d}_1} = 0 , \; l_1p_1 = lp , \; m_1n_1 = mn .$$

103. Consider, in particular, the case where $p = 0$; the relation

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0 ,$$

here becomes

$$l = -\frac{ag'}{bf'}m - \frac{a'h}{c'f}n .$$

The equation in θ is

$$(cc'm + bb'n)\theta + cb'n\theta^2 + bc'm = 0\ ,$$

viz., this is

$$(c\theta + c'm)(b'n\theta + b) = 0\ ,$$

giving

$$\theta = -\frac{b}{c}\ ,\quad m_1 = -\frac{bn}{c}\ ,\quad n_1 = -\frac{cm}{b}\ ,$$

or else

$$\theta = -\frac{c'm}{b'n}\ ,\quad m_1 = -\frac{c'm}{b'}\ ,\quad n_1 = -\frac{b'n}{c'}\ .$$

Since in the present case $l_1 p_1 = 0$, we have either $l_1 = 0$, or else $p_1 = 0$, and as might be anticipated, the two values of θ correspond to these two cases respectively, viz., proceeding to find the values of l_1, p_1, the completed systems are

$$\theta = -\frac{b}{c}\ , l_1 = \frac{a}{bcf'}\left(cc'm - bb'n\right)\ , m_1 = -\frac{bn}{c}\ , n_1 = -\frac{cm}{b}\ , p_1 = 0\ .$$

$$\theta = -\frac{c'm}{b'n}\ , l^{\backprime}_1 = 0 \qquad\qquad , m^{\backprime}_1 = -\frac{c'm}{b'}\ , n^{\backprime}_1 = -\frac{b'n}{c'}\ , p^{\backprime}_1 = \frac{a'}{b'c'f}\left(cc'm - bb'n\right)\ ,$$

so that for the first system we have

$$\frac{l_1}{\mathrm{a}_1} + \frac{m_1}{\mathrm{b}_1} + \frac{n_1}{\mathrm{c}_1} = 0\ , m_1 n_1 = mn\ , -l\mathsf{A} + m\mathsf{B} + n\mathsf{C} = -l_1\mathsf{A}_1 + m_1\mathsf{B}_1 + n_1\mathsf{C}_1\ ,$$

and for the second system

$$\frac{m^{\backprime}_1}{\mathrm{b}_1} + \frac{n^{\backprime}_1}{\mathrm{c}_1} + \frac{p^{\backprime}_1}{\mathrm{d}_1} = 0\ , m^{\backprime}_1 n^{\backprime}_1 = mn\ , -l\mathsf{A} + m\mathsf{B} + n\mathsf{C} = -p^{\backprime}_1\mathsf{D}_1 + m^{\backprime}_1\mathsf{B}_1 + n^{\backprime}_1\mathsf{C}_1\ .$$

104. The whole of the foregoing investigation would have assumed a more simple form if the circular co-ordinates had been taken with reference to the centre of the circle $ABCD$ as origin, and the radius of this circle been put $= 1$; we should then have $a' = \frac{1}{a}$, &c., and consequently

$$a' = -\frac{1}{\beta\gamma}a\ , b' = -\frac{1}{\gamma\alpha}b\ , c' = -\frac{1}{\alpha\beta}c_1\ , f' = -\frac{1}{\alpha\delta}f\ , g' = -\frac{1}{\beta\delta}g\ , h' = -\frac{1}{\gamma\delta}h\ ;$$

but the symmetrical relation of the circles $ABCD$ and $A_1B_1C_1D_1$ would not have been so clearly shown.

I will however give the investigation in this simplified form, for the identity $-l\mathsf{A} + m\mathsf{B} + n\mathsf{C} = -l_1\mathsf{A} + m_1\mathsf{B} + n_1\mathsf{C}$; viz., in this case we have

$$\frac{l}{\alpha} = -\frac{m(\beta-\gamma)(\beta-\delta)}{\beta(\gamma-\alpha)(\alpha-\delta)} - \frac{n(\beta-\gamma)(\gamma-\delta)}{\gamma(\alpha-\beta)(\alpha-\gamma)}\ ,$$

and the identity to be satisfied is

$$\begin{aligned} - \ l(\xi - \alpha z)\left(\eta - \frac{1}{\alpha}z\right) &= - \ l_1(\xi - \alpha z)\left(\eta - \frac{1}{\delta}z\right) \\ + \ m(\xi - \beta z)\left(\eta - \frac{1}{\beta}z\right) &\quad + m_1(\xi - \beta z)\left(\eta - \frac{1}{\gamma}z\right) \\ + \ n(\xi - \gamma z)\left(\eta - \frac{1}{\gamma}z\right) &\quad + n_1(\xi - \gamma z\left(\eta - \frac{1}{\beta}z\right); \end{aligned}$$

writing $\xi = az$, $\eta = \frac{1}{\beta}z$, we find m_1, and writing $\xi = az$, $\eta = \frac{1}{\gamma}z$, we find n_1, and it is then easy to obtain the value of l_1, viz., the results are

$$\frac{l_1}{\delta} = \frac{m}{\beta}\frac{(\alpha - \beta)(\beta - \gamma)}{(\gamma - \alpha)(\alpha - \delta)} + \frac{n}{\gamma}\frac{(\beta - \gamma)(\gamma - \alpha)}{(\alpha - \beta)(\alpha - \delta)}, \quad m_1 = - n\frac{\gamma - \alpha}{\alpha - \beta}, \quad n_1 = - m\frac{\alpha - \beta}{\gamma - \alpha},$$

and therefore $m_1 n_1 = mn$; it may be added that we have

$$- \frac{l_1}{\delta} = \frac{\beta - \gamma}{\alpha - \delta}\left(\frac{m_1}{\gamma} + \frac{n_1}{\delta}\right),$$

viz., this is the form assumed by the equation $\frac{l_1}{a_1} + \frac{m_1}{b_1} + \frac{n_1}{c_1} = 0$.

PART III. (Nos. 105 to 157.)—ON THE THEORY OF FOCI.

Explanation of the General Theory—Art. Nos. 105 to 110.

105. If from a focus of a conic we draw two tangents to the curve, these pass respectively through the two circular points at infinity, and we have thence the generalised definition of a focus as established by PLÜCKER, viz., in any curve a focus is a point such that the lines joining it with the two circular points at infinity are respectively tangents to the curve; or, what is the same thing, if from each of the circular points at infinity, say from the points I, J, tangents are drawn to the curve, the intersections of each tangent from the one point with each tangent from the other point are the foci of the curve. A curve of the class n has thus in general n^2 foci. It is to be added that, as in the conic the line joining the points of contact of the two tangents from a focus is the directrix corresponding to that focus, so in general the line joining the points of contact of the tangents from the focus through the points I, J respectively is the directrix corresponding to the focus in question.

106. A circular point at infinity I or J, may be an ordinary or a singular point on the curve, and the tangent at this point then counts, or, in the case of a multiple point, the tangents at this point count a certain number of times, say q times, among the tangents which can be drawn to the curve from the point; the number of the remaining tangents is thus $= n - q$. In particular, if the circular point at infinity be an ordinary point, then the tangent counts twice, or

we have $q = 2$; if it be a node, each of the tangents count twice, or $q = 4$; if it be a cusp, the tangent counts three times, or $q = 3$. Similarly, if the other circular point at infinity be an ordinary or a singular point on the curve, the tangent or tangents there count a certain number of times, say q' times, among the tangents to the curve from this point; the number of the remaining tangents is thus $= n - q'$. And if as usual we disregard the tangents at the two points I, J respectively, and attend only to the remaining tangents, the number of the foci is $= (n - q)(n - q')$.

107. Among the tangents from the point I or J there may be a tangent which, either from its being a multiple tangent (that is, a tangent having ordinary contact at two or more distinct points), or from being an osculating tangent at one or more points, counts a certain number of times, say r, among the tangents from the point in question. Similarly, if among the tangents from the other point J or I, there is a tangent which counts r' times, then the foci are made up as follows, viz. we have—

Intersections of the two singular tangents counting as	$r'r$ foci.
Intersections of the first singular tangent with each of the ordinary tangents from the other circular point at infinity, as	$(n-q'-r')r$,,
Do. for second singular tangent, . . .	$(n-q-r)r'$,,
Intersections of the ordinary tangents, . .	$(n-q-r)(n-q'-r')$,,
Giving together the	$(n-q)(n-q')$ foci:

and the like observation applies to the more general case where the tangents from each of the points I, J include more than one singular tangent.

108. There is yet another case to be considered; the line infinity may be an ordinary or a singular tangent to the curve: assuming that it counts s times among the tangents from either of the circular points at infinity, the numbers of the remaining tangents are $n - q - s$, $n - q' - s$ from the two points I, J respectively, and the number of foci is $= (n - q - s)(n - q' - s)$.

109. In the case of a real curve the two points I, J are related in the same manner to the curve, and we have therefore $q = q'$; the singular tangents (if any) from the two points respectively being the same as well in character as in number. Writing $n - q - s = n - q' - s, = p$, and not for the present attending to the case of singular tangents, I shall assume that the number of tangents to the curve from each of the two points is $= p$; the number of foci is thus $= p^2$; and to each focus there corresponds a directrix, viz., this is the line through the points of contact of the tangents from the focus to the two points I, J respectively.

110. Consider any two foci A, B not *in lineâ* with either of the points I, J, then joining these with the points I, J, and taking A_1, B_1 the intersections of AI, BJ and of AJ, BI (A_1, B_1 being therefore by a foregoing definition the anti-

points of (A, B)), then A_1, B_1 are, it is clear, foci of the curve. We may out of the p^2 foci select, and that in $1.2\,..\,p$ different ways, a system of p foci such that no two of them lie *in lineâ* with either of the points I, J; and this being so, taking the anti-points of each of the $\tfrac{1}{2}p(p-1)$ pairs out of the p foci, we have, inclusively of the p foci, in all $p+2\,.\,\tfrac{1}{2}p(p-1)$, that is p^2 foci, the entire system of foci.

On the Foci of Conics—Art. Nos. 111 to 117.

111. A conic is a curve of the class 2, and the number of foci is thus $= 4$. Taking as foci any two points A, B, the remaining two foci will be the anti-points A_1, B_1. In order that a given point A may be a focus, the conic must touch the lines AI, AJ; similarly, in order that a given point B may be a focus, the conic must touch the lines BI, BJ; the equation of a conic having the given points A, B for foci contains therefore a single arbitrary parameter.

112. In the case, however, of the parabola the curve touches the line infinity; there is consequently from each of the points I, J only a single tangent to the curve, and consequently only one focus: the parabola having a given point A for its focus is a conic touching the line infinity and the lines AI, AJ, or say the three sides of the triangle AIJ; its equation contains therefore two arbitrary parameters.

113. Returning to the general conic, there are certain trizomal forms of the focal equation, not of any great interest, but which may be mentioned. Using circular co-ordinates, and taking $(\alpha, \alpha', 1)$ and $(\beta, \beta', 1)$ for the co-ordinates of the given foci A, B respectively, the conic touches the lines $\xi-\alpha z=0$, $\eta-\alpha' z=0$, $\xi-\beta z=0$, $\eta-\beta' z=0$; the equation of a conic touching the first three lines is

$$\sqrt{l(\xi-\alpha z)}+\sqrt{m(\xi-\beta z)}+\sqrt{n(\eta-\alpha' z)}=0,$$

where l, m, n are arbitrary, and it is easy to obtain, in order that the conic may touch the fourth line $\eta-\beta' z=0$, the condition

$$n=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l).$$

114. In fact, n having this value, the equation gives

$$l(\xi-\alpha z)+m(\xi-\beta z)+2\sqrt{lm(\xi-\alpha z)(\xi-\beta z)}=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)\big(\eta-\beta' z+(\beta'-\alpha')z\big),$$

and taking over the term $\dfrac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\beta'-\alpha')z,\ =(\beta-\alpha)(m-l)z,$

this gives

$$l(\xi-\beta z)+m(\xi-\alpha z)+2\sqrt{lm(\xi-\alpha z)(\xi-\beta z)}=-\frac{\beta-\alpha}{\beta'-\alpha}(m-l)(\eta-\beta' z),$$

which puts in evidence the tangent $\eta - \beta' z$. It is easy to see that the equation may be written in any one of the four forms

$$\sqrt{l(\xi - \alpha z)} + \sqrt{m(\xi - \beta z)} + \sqrt{-\frac{\beta - \alpha}{\beta' - \alpha'}(m - l)(\eta - \alpha' z)} = 0\,,$$

$$\sqrt{m(\xi - \alpha z)} + \sqrt{l(\xi - \beta z)} + \sqrt{-\frac{\beta - \alpha}{\beta' - \alpha'}(m - l)(\eta - \beta' z)} = 0\,,$$

$$\sqrt{l(\eta - \alpha' z)} + \sqrt{m(\eta - \beta' z)} + \sqrt{-\frac{\beta - \alpha'}{\beta - \alpha}(m - l)(\xi - \alpha z)} = 0\,,$$

$$\sqrt{m(\eta - \alpha' z)} + \sqrt{l(\eta - \beta' z)} + \sqrt{-\frac{\beta' - \alpha'}{\beta - \alpha}(m - l)(\xi - \beta z)} = 0\,.$$

viz., in forms containing any three of the four radicals $\sqrt{\xi - \alpha z}$, $\sqrt{\xi - \beta z}$, $\sqrt{\eta - \alpha' z}$, $\sqrt{\eta - \beta' z}$. The conic is thus expressed as a trizomal curve, the zomals being each a line, viz., they are any three out of the four focal tangents; the order of the curve, as deduced from the general expression $2^{\nu-2}r$, is $= 2$; so that there is here no depression of order.

115. But the ordinary form of the focal equation is a more interesting one; viz., A, B being as usual the squared distances of the current point from the two given foci respectively, say

$$\mathsf{A} = (\xi - \alpha z)\,\eta - \alpha' z)\,,$$
$$\mathsf{B} = (\xi - \beta z)\,\eta - \beta' z)\,,$$

then $2a$ being an arbitrary parameter, the equation is

$$2az + \sqrt{\mathsf{A}} + \sqrt{\mathsf{B}} = 0\,,$$

viz., the equation is here that of a trizomal curve, the zomals being curves of the second order, that is, the zomals are ($z^2 = 0$) the line infinity twice, and the line-pairs AI, AJ and BI, BJ respectively: the general expression $2^{\nu-2}r$ gives therefore the order $= 4$; but in the present case there are two branches, viz., the branches

$$2az + \sqrt{\mathsf{A}} - \sqrt{\mathsf{B}} = 0\,,\ 2az - \sqrt{\mathsf{A}} + \sqrt{\mathsf{B}} = 0\,,$$

each ideally containing ($z = 0$) the line infinity; the curve contains therefore ($z^2 = 0$) the line infinity twice, and omitting this factor the order is $= 2$, as it should be.

116. To express the equation by means of the other two foci A_1, B_1, writing the equation under the form

$$\mathsf{A} + \mathsf{B} + 2\sqrt{\mathsf{A}\mathsf{B}} - 4a^2z^2 = 0\,,$$

and then if A_1, B_1 are the squared distances of the current point from A_1, B_1 respectively, we have (*ante*, No. 65).

$$\mathsf{A}\mathsf{B} = \mathsf{A}_1\mathsf{B}_1\,,$$
$$\mathsf{A} + \mathsf{B} - \mathsf{A}_1 - \mathsf{B}_1 = kz^2\,,$$

where k is the squared distance of the foci $A, B, = 4a^2e^2$ suppose: whence putting $a^2(1 - e^2) = b^2$, the equation becomes

$$\mathsf{A}_1 + \mathsf{B}_1 + 2\sqrt{\mathsf{A}_1\mathsf{B}_1} - 4b^2z^2 = 0\,,$$

that is

$$\sqrt{\mathsf{A}_1} + \sqrt{\mathsf{B}_1} + 2bz = 0\,,$$

which is the required new form. It is hardly necessary to remark that the equation $2az + \sqrt{\mathsf{A}} + \sqrt{\mathsf{B}} = 0$, putting therein $z = 1$, and expressing A, B in rectangular co-ordinates measured along the axes, is the ordinary focal equation $2a = \sqrt{(x - ae)^2 + y^2} + \sqrt{(x + ae)^2 + y^2}$.

117. I remark that the equation $2az + \sqrt{\mathsf{A}} + \sqrt{\mathsf{B}} = 0$, gives rise to $4a^2z^2 + \mathsf{A} - \mathsf{B} + 4az\sqrt{\mathsf{A}} = 0$, but here $\mathsf{A} - \mathsf{B} = -4aexz$, so that the equation contains $z = 0$, and omitting this it becomes $(az - ex) + \sqrt{\mathsf{A}} = 0$, a bizomal form, being a curve of the order $= 2$, as it should be; this is in fact the ordinary equation in regard to a focus and its directrix.

Theorem of the Variable Zomal as applied to a Conic—Art. Nos. 118 to 123.

118. The equation $2kz + \sqrt{\mathsf{A}^\circ} + \sqrt{\mathsf{B}^\circ} = 0$ is in like manner that of a conic: in fact, this would be a curve of the order $= 4$, but there are as before the two branches $2kz + \sqrt{\mathsf{A}^\circ} - \sqrt{\mathsf{B}^\circ} = 0$, $2kz - \sqrt{\mathsf{A}^\circ} + \sqrt{\mathsf{B}^\circ} = 0$, each ideally containing $(z = 0)$ the line infinity, and the order is thus reduced to be $= 2$. Each of the circles $\mathsf{A}^\circ = 0$, $\mathsf{B}^\circ = 0$ is a circle having double contact with the conic (this of course implies that the centre of the circle is on an axis of the conic). We may if we please start from the form $2kz + \sqrt{\mathsf{A}} + \sqrt{\mathsf{B}} = 0$, and then by means of the theorem of the variable zomal introduce into the equation one, two, or three such circles.

119. It is in this point of view that I will consider the question, viz., adapting the formula to the case of the ellipse, and starting from the form

$$2az + \sqrt{(x - aez)^2 + y^2} + \sqrt{(x + aez)^2 + y^2} = 0\,,$$

the equation of the variable zomal or circle of double contact may be taken to be

$$\frac{4a^2z^2}{-2} + \frac{(x - aez)^2 + y^2}{1 - q} + \frac{(x + aez)^2 + y^2}{1 + q} = 0\,,$$

where q is an arbitrary parameter; writing for greater simplicity $z = 1$, and reducing, the equation is

$$(x - qae)^2 + y^2 = b^2(1 - q^2)\,.$$

120. If $q < 1$, then writing $q = \sin\theta$, we obtain the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

as the envelope of the variable circle

$$(x - ae\sin\theta)^2 + y^2 = b^2\cos^2\theta \ ,$$

viz., of a circle having its centre on the major axis at a distance $= ae \sin\theta$ from the centre, and its radius $= b\cos\theta$. (I notice, in passing, that this gives in practice a very convenient graphical construction of the ellipse.) It may be remarked that for $\theta = \pm\sin^{-1} e$, the circle becomes

$$\left(x \pm \left(a - \frac{b^2}{a}\right)\right)^2 + y^2 = \frac{b^4}{a^2} \ ,$$

viz., this is the circle of curvature at one or other of the extremities of the major axis; as θ passes from 0 to $\pm\sin^{-1} e$ we have a series of real circles, which, by their continued intersection, generate the ellipse; as θ increases from $\theta = \pm\sin^{-1} e$ to $\pm 90^\circ$, the circles continue real, but the consecutive circles no longer intersect in any real point,—and ultimately for $\theta = \pm 90^\circ$, the circles become evanescent at the two foci respectively.

121. In the case $q > 1$, we have a real representation of

$$(x - qae)^2 + y^2 + b^2(q^2 - 1) \ ,$$

as the squared distance of the point (x, y) from a point $(X, 0, Z)$ out of the plane of the figure, viz., putting this

$$= (x - X)^2 + y^2 + Z^2 \ ,$$

we have

$$qae = X, \ Z^2 = b^2(q^2 - 1) \ ,$$

whence

$$Z^2 = b^2\left(\frac{X^2}{a^2e^2} - 1\right) ,$$

or what is the same thing,

$$\frac{X^2}{a^2 - b^2} - \frac{Z^2}{b^2} = 1 \ ;$$

that is, the locus is the focal hyperbola, viz., a hyperbola in the plane of zx, having its vertices at the foci, and its foci at the vertices of the ellipse.

122. If instead of the form first considered, we start from the trizomal form

$$2bz + \sqrt{x^2 + (y - aeiz)^2} + \sqrt{x^2 + (y + aeiz)^2} = 0 \ ,$$

then we have the zomal or circle of double contact under the form

$$x^2 + (y - qaei)^2 = a^2(1 - q^2) \ ;$$

or putting herein $q = -i\tan\phi$, this is,

$$x^2 + (y - ae\tan\varphi)^2 = a^2\sec^2\varphi \ ;$$

so that we have the ellipse as the envelope of a variable circle having its centre on the minor axis of the ellipse, distance from the centre $= ae\tan\phi$, and radius

$= a \sec\phi$. This is, in fact, Gergonne's theorem, according to which the ellipse is the secondary caustic or orthogonal trajectory of rays issuing from a point and refracted at a right line into a rarer medium. It is to be remarked that for $\tan\phi = \pm \frac{ae}{b}$, the equation of the circle is

$$x^2 + \left((y \pm (b - \frac{a^2}{b})\right)^2 = \frac{a^4}{b^2},$$

viz., this is the circle of curvature at one or other extremitity of the minor axis; from $\phi = 0$ to $\phi = \pm \tan^{-1}\frac{ae}{b}$, the intersections of the consecutive circles are real, and give the entire real ellipse; from $\phi = \pm \tan^{-1}\frac{ae}{b}$, to $\phi = \pm 90^\circ$, the circles are still real, but the intersections of consecutive circles are imaginary.

123. If in the equation of the generating circle we interchange x, y, a, b, the equation becomes

$$(x - aei \tan\varphi)^2 + y^2 = b^2 \sec^2\varphi,$$

which is (as it should be) equivalent to the former equation

$$(x - ae \sin\theta)^2 + y^2 = b^2 \cos\theta,$$

the identity being established by means of the equation

$$\cos\theta = \frac{1}{\cos\varphi}, \text{ and } \therefore \sin\theta = i \tan\varphi, \tan\theta = i \sin\varphi,$$

which is Jacobi's imaginary transformation in the theory of Elliptic Functions.

Foci of the Circular Cubic and the Bicircular Quartic—Art. Nos. 124 to 126.

124. For a cubic curve, the class is in general = 6, and the number of the foci is = 36. But a specially interesting case is that of a circular cubic, viz., a cubic passing through each of the circular points at infinity. Here, at each of the circular points at infinity, the tangent at this point reckons twice among the tangents to the curve from the point; the number of the remaining tangents is thus = 4, and the number of the foci is = 16. If from any two points whatever on the curve tangents be drawn to the curve, then the two pencils of tangents are, and that in four different ways, homologous to each other, viz., if the tangents of the first pencil are (1, 2, 3, 4), and those of the second pencil, taken in a proper order, are (1′, 2′, 3′, 4′), then we have (1, 2, 3, 4) homologous with each of the arrangements (1′, 2′, 3′, 4′), (2′, 1′, 4′, 3′), (3′, 4′, 1′, 2′),

$(4', 3', 2', 1')$. And in each case the intersections of the four corresponding tangents lie on a conic passing through the two given points on the curve.*

125. Hence taking the points on the curve to be the circular points at infinity, we have the sixteen foci lying in fours upon four different circles—that is, we have four tetrads of concyclic foci. Let any one of these tetrads be A, B, C, D, then if

Anti-points of $(B, C)\ (A, D)$ are $(B_1, C_1), (A_1, D_1)$,
" $(C, A)\ (B, D)$ " $(C_2, A_2), (B_2, D_2)$,
" $(A, B)\ (C, D)$ " $(A_3, B_3), (C_3, D_3)$,

the four tetrads of concyclic foci are

$$\begin{array}{llll} A\ , & B\ , & C\ , & D\ ; \\ A_1, & B_1, & C_1, & D_1\ ; \\ A_2, & B_2, & C_2, & D_2\ ; \\ A_3, & B_3, & C_3, & D_3\ . \end{array}$$

It is to be observed that if A, B, C, D are any four points on a circle, then if, as above, we pair these in any manner, and take the anti-points of each pair, the four anti-points lie on a circle, and thus the original system A, B, C, D, of four points on a circle, leads to the remaining three systems of four points on a circle. The theory is in fact that already discussed *ante*, No. 72 *et seq.*

126. The preceding theory applies without alteration to the bicircular quartic, viz., the quartic curve which has a node at each of the circular points at infinity. The class is here $= 8$, but among the tangents from a node each of the two tangents at the node is to be reckoned twice, and the number of the remaining tangents is $= 4$: the number of foci is $= 16$. And, by the general theorem that in a binodal quartic the pencils of tangents from the two nodes respectively are homologous, the sixteen foci are related to each other precisely in the manner of the foci of the circular cubic. The latter is in fact a particular case of the former, viz., the bicircular quartic may break up into the line infinity, and a circular cubic.

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* It may be remarked that if the equation of the first pencil of lines be

$$(x - ay)(x - by)(x - cy)(x - dy) = 0\ ,$$

and that of the second pencil

$$(z - aw)(z - bw)(z - cw)(z - dw) = 0\ ,$$

then the equations of four conics are

$$xw - yz = 0\ ,$$
$$(a + d - b - c)\,xz + (bc - ad)(xw + yz) + \big(ad(b + c) - bc(a + d)\big)yw = 0\ ,$$
$$(b + d - c - a)\,xz + (ca - bd)(xw + yz) + \big(bd(c + a) - ca(b + d)\big)yw = 0\ ,$$
$$(c + d - a - b)\,xz + (ab - cd)(xw + yz) + \big(cd(a + b) - ab(c + d)\big)yw = 0\ .$$

Centre of the Circular Cubic, and Nodo-Foci, &c. of the Bicircular Quartic—Art. Nos. 127 to 129.

127. The tangents at I, J have not been recognised as tangents from I, J, giving by their intersection a focus, but it is necessary in the theory to pay attention to the tangents in question. It is clear that these tangents are in fact asymptotes—viz., in the case of the circular cubic they are the two imaginary asymptotes of the curve, and in the case of a bicircular quartic, the two pairs of imaginary parallel asymptotes; but it is convenient to speak of them as the tangents at I, J.

128. In the case of a circular cubic, the tangents at I and J meet in a point which I call the centre of the curve, viz., this is the intersection of the two imaginary asymptotes.

129. In the case of a bicircular quartic, the two tangents at I and the two tangents at J meet in four points, which (although not recognising them as foci) I call the nodo-foci; these lie in pairs on two lines, diagonals of the quadrilateral formed by the four tangents (the third diagonal is of course the line IJ), which diagonals I call the "nodal axes;" and the point of intersection of the two nodal axes is the "centre" of the curve. The nodo-foci are four points, two of them real, the other two imaginary, viz., they are two pairs of anti-points, the lines through the two pairs respectively being, of course, the nodal axes; these are consequently real lines bisecting each other at right angles in the centre (with the relation $1 : i$ between the distances). The centre may also be defined as the intersection of the harmonic of IJ in regard to the tangents at I, and the harmonic of this same line in regard to the tangents at J. Speaking of the tangents as asymptotes, the nodo-foci are the angles of the rhombus formed by the two pairs of parallel asymptotes; the nodal axes are the diagonals of this rhombus, and the centre is the point of intersection of the two diagonals; as such it is also the intersection of the two lines drawn parallel to and midway between the lines forming each pair of parallel asymptotes.

Circular Cubic and Bicircular Quartic; the Axial or Symmetrical Case—Art. No. 130.

130. In a circular cubic or bicircular quartic, the pencil of the tangents from I and that of the tangents through J, considered as corresponding to each other in some one of the four arrangements, may be such that the line IJ considered as belonging to the two pencils respectively shall correspond to itself, and when this is so, the four foci, A, B, C, D, which are the intersections of the corresponding tangents in question, will lie in a line (viz., the conic which exists in the general case will break up into a line-pair consisting of the line IJ and another line). The line in question may be called the focal axis; it will presently be shown that in the case of the circular cubic it passes through the centre, and that in the case of the bicircular quartic it not only passes through the centre, but

coincides with one or other of the nodal axes, viz., with that passing through the real or the imaginary nodo-foci; that is, the curve may have on the focal axis two real or else two imaginary nodo-foci. The focal axis contains, as has been mentioned, four foci—the remaining twelve foci are situate symmetrically, six on each side of the focal axis, the arrangement of the sixteen foci being as mentioned *ante*, No. 81 *et seq.*; the focal axis is in fact an axis of symmetry of the curve, and if preferred it may be named the axis of symmetry, transverse axis, or simply the axis. And the curve (circular cubic, or bicircular quartic) is in this case a "symmetrical" or "axial" curve.

Circular Cubic and Bicircular Quartic: Singular Forms—Art. Nos. 131 to 140.

131. The circular cubic may have a node or a cusp. If this were at one of the points I, J the curve would be imaginary, and I do not attend to the case; and for the same reason, for the bicircular quartic I do not attend to the case where *one* of the points I, J is a cusp. There remain then for the circular cubic and for the bicircular quartic the cases where there is a node or a cusp at a real point of the curve; and for the bicircular quartic the case where each of the points I, J is a cusp—in general the curve has no other node or cusp, but it may besides have a node or cusp at a real point thereof.

132. I consider first the case of the bicircular quartic where each of the points I, J is a cusp. The curve is in this case of necessity symmetrical*—it is in fact a Cartesian; viz., the Cartesian may be taken by definition to be a quartic curve having a cusp at each of the circular points at infinity. But in this case, as distinguished from the general case of the bicircular quartic, there is an essential degeneration of all the focal properties, and it is necessary to explain what these become. The centre is evidently the intersection of the cuspidal tangents; the nodo-foci (so far as they can be said to exist) coalesce with the centre, and they do not in so coalescing determine any definite directions for the nodal axes; that is, there are no nodal axes, and the only theorem in regard to the focal axis or axis of symmetry is, that it passes through the centre. Of the four tangents through the point I, one has come to coincide with the line IJ; and similarly, of the four tangents through the point J one has come to coincide with the line JI: there remain only three tangents through I and three tangents through J, and these by their intersections determine nine foci—viz., three foci A, B, C on the axis, and besides (B_1, C_1) the anti-points of (B, C) : (C_2, A_2) the anti-points of (C, A) and (A_3, B_3) the anti-points of (A, B).

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* It will appear, *post* Nos. 161–164, that if starting with three given points as the foci of a bicircular quartic, we impose the condition that the nodes at I, J shall be each of them a cusp, then either the quartic will be the circle through the three points taken twice, in which case the assumed focal property of the given three points disappears altogether, or else the three points must be *in lineâ*, or the curve be symmetrical, that is, a Cartesian.

133. The remaining seven foci have disappeared, viz., we may consider that one of them has gone off to infinity on the focal axis, and that three pairs of foci have come to coincide with the points I, J respectively. The circle O (as in the general case of a symmetrical quartic) has become a line, the focal axis; the circles R, S, T (contrary to what might at first sight appear) continue to be determinate circles, viz., these have their centres at A, B, C respectively, and pass through the points (B_1, C_1), (C_2, A_2), and (A_3, B_3) respectively, see *ante*, No. 83. But on each of these circles we have not more than two proper foci, and it is only on the axis as representing the circle O that we have three proper foci, the axial foci A, B, C: in regard hereto it is to be remarked that the equation of the curve can be expressed not only by means of these three foci in the form $\sqrt{l\mathsf{A}} + \sqrt{m\mathsf{B}} + \sqrt{n\mathsf{C}} = 0$; but by means of any two of them in the form $\sqrt{l\mathsf{A}} + \sqrt{m\mathsf{B}} + K = 0$, where K is a constant, or, what is the same thing (z being introduced for homogeneity in the expressions of A and B respectively), in the form $\sqrt{l\mathsf{A}} + \sqrt{m\mathsf{B}} + Kz^2 = 0$.

134. Using for the moment the expression "twisted" as opposed to symmetrical—(viz., the curve is twisted when there is not any axis of symmetry but the foci lie only on circles)—then the classification is

Circular Cubics, twisted,
„ „ symmetrical,
Bicircular Quartics, twisted,
„ „ symmetrical, $\left\{\begin{array}{l}\text{Ordinary,}\\ \text{Bicuspidal} = \text{Cartesian,}\end{array}\right.$

and each of these kinds may be general, nodal, or cuspidal—viz., for the two last mentioned kinds there may be a node or a cusp at a real point of the curve.

135. In the case of a node, say the point N; first if the curve (circular cubic or bicircular quartic) be twisted—then of the four foci A, B, C, D we have two, suppose B and C, coinciding with N; and the sixteen foci are as follows, viz.

$B,\ C,\ A,\ D$	are	N, N, A, D;
B_1, C_1, A_1, D_1	„	N, N, Anti-pts. of (A, D);
C_2, A_2, B_2, D_2	„	Anti-pts. of (N, A), Anti-pts. of (N, D);
A_2, B_3, C_3, D_3	„	Do. do.

viz., we have the points (A, D) each once, the node N four times, the anti-points of (A, D) once, and the anti-points of (N, A) and of (N, D), each pair twice. But properly there are only four foci, viz., the points A, D and their anti-points. The circle O subsists as in the general case, and so does the circle R (BC, AD), viz., this has for centre the intersection of the line AD by the tangent at N to the circle O, and it passes through the point N, of course cutting the circle O at right angles: the circles S and T each reduce themselves each to the point N considered as an evanescent circle, or what is the same thing to the line-pair NI, NJ.

136. The case is nearly the same if the curve be symmetrical, but in the case of the bicircular quartic excluding the Cartesian: viz., we have on the axis the foci B, C coinciding at N, and the other two foci A, D; the sixteen foci are as above—and the circle R is determined by the proper construction as applied to the case in hand, viz., the centre R is the intersection of the axis by the radical axis of the point N (considered as an evanescent circle) and the circle on AD as diameter; that is $\overline{RN}^2 = RA \,.\, RD$. And the circles S and T reduce themselves each to the point N considered as an evanescent circle.

137. Next if we have a cusp, say the point K: first if the curve (circular cubic or bicircular quartic) be twisted—then of the four foci A, B, C, D, three, suppose A, B, C, coincide with K; and the sixteen foci are as follows, viz.,

B, C, A, D	are	K, K, K, D,	
B_1, C_1, A_1, D_1	„	K, K,	Anti-points of (K, D),
C_2, A_2, B_2, D_2	„	Do.	do.
A_3, B_3, C_3, D_3	„	Do.	do.

viz., we have the point D once, the point K nine times, and the anti-points of K, D three times. But properly the point D is the only focus. The circle O is, it would appear, *any* circle through K, D, but possibly the particular circle which touches the cuspidal tangent may be a better representative of the circle O of the general case—the circles R, S, T reduce themselves each to the point K considered as an evanescent point.

138. The like is the case if the curve be symmetrical, but in the case of the bicircular quartic excluding the Cartesian; the circle O is here the axis, which is in fact the cuspidal tangent.

139. For the Cartesian, if there is a node N; then of the three foci A, B, C, two, suppose B and C, coincide with N; the nine foci are A once, N four times, and the anti-points of N, A twice: but properly the point A is the only focus. And if there be a cusp K; then all the three foci A, B, C coincide with K; and the nine foci are K nine times; but in fact there is no proper focus.

140. A circular cubic cannot have two nodes unless it break up into a line and circle; and similarly a bicircular quartic cannot have two nodes (exclusive of course of the points I, J) unless it break up into two circles; the last-mentioned case will be considered in the sequel in reference to the problem of tactions.

As to the Analytical Theory for the Circular Cubic and the Bicircular Quartic respectively—

Art. No. 141.

141. It may be remarked in regard to the analytical theory about to be given, that although the investigation is very similar for the circular cubic and for the bicircular quartic, yet the former cannot be deduced from the latter case. In fact if for the bicircular quartic, using a form somewhat more general than that

which is ultimately adopted, we suppose that for the two nodes respectively $(\xi = 0,\ z = 0)$ and $(\eta = 0,\ z = 0)$, then if $l\xi + mz = 0$, $l'\xi + m'z = 0$, $n\eta + pz = 0$, $n'\eta + p'z = 0$ are the tangents at the two nodes respectively, the equation will be

$$(l\xi + mz)(l'\xi + m'z)(n\eta + pz)(n'\eta + p'z) + ez^2\xi\eta + z^3(a\xi + b\eta) + cz^4 = 0,$$

and if (in order to make this equation divisible by z, and the curve so to break up into the line $z = 0$ and a cubic) we write $l = 0$ or $n = 0$, then the curve will indeed break up as required, but we shall have, not the general cubic through the two points $(\xi = 0,\ z = 0)$, $(\eta = 0,\ z = 0)$, but in each case a nodal cubic, viz., if $l = 0$ there will be a node at the point $(\eta = 0,\ z = 0)$, and if $n = 0$ a node at the point $(\xi = 0,\ z = 0)$.

Analytical Theory for the Circular Cubic—Art. Nos. 142 to 144.

142. I consider then the two cases separately; and first the circular cubic. The equation may be taken to be

$$\xi\eta(p\xi + q\eta) + ez\xi\eta + z^2(a\xi + b\eta + cz^3) = 0,$$

or what is the same thing

$$\xi\eta(p\xi + q\eta + ez) + z^2(a\xi + b\eta + cz) = 0,$$

viz. (ξ, η, z) being any co-ordinates whatever, this is the general equation of a cubic passing through the points $(\xi=0, z=0)$, $(\eta=0, z=0)$, and at these points touched by the lines $\xi = 0$, $\eta = 0$ respectively. And if $(\xi, \eta, z = 1)$ be circular co-ordinates, then we have the general equation of a circular cubic having the lines $\xi = 0$, $\eta = 0$ for its asymptotes, or say the point $\xi = 0$, $\eta = 0$ for its centre; the equation of the remaining asymptote is evidently $p\xi + q\eta + ez = 0$; to make the curve real we must have (p, q) and (a, b) conjugate imaginaries, e and c real.

143. Taking in any case the points I, J to be the points $\xi = 0$, $z = 0$ and $\eta = 0$, $z = 0$ respectively, for the equation of a tangent from I write $p\xi = \theta z$; then we have

$$\theta\eta(\theta z + q\eta + ez) + z(a\theta z + bp\eta + cpz) = 0,$$

that is

$$z^2(a\theta + cp) + \eta z(\theta^2 + e\theta + bp) + \eta^2 . q\theta = 0,$$

and the line will be a tangent if only

$$(\theta^2 + e\theta + bp)^2 - 4q\theta(a\theta + cp) = 0,$$

that is, the four tangents from I are the lines $p\xi = \theta z$, where θ is any root of this equation. Similarly the four tangents from J are the lines $q\eta = \phi z$, where ϕ is any root of the equation

$$(\varphi^2 + e\varphi + aq)^2 - 4p\varphi(b\varphi + cq) = 0,$$

writing the two equations under the forms

$$\left\{\begin{array}{l} 6, \\ 3e, \\ c^2 + 2bp - 4aq, \\ 3ebp \qquad - 6cpq, \\ 6b^2p^2, \end{array}\right\} (\theta,\ 1)^4 = 0\,, \quad \left\{\begin{array}{l} 6, \\ 3e, \\ c^2 + 2aq - 4bp, \\ 3eaq \qquad - 3cpq, \\ 6a^2q^2, \end{array}\right\} (\phi, 1)^4 = 0\,,$$

the equations have the same invariants; viz., for the first equation the invariants are easily found to be

$$\begin{aligned} I &= \ \ 3(c^2 - 4bp - 4aq)^2 + 72(ce - 2ab)pq\,, \\ J &= -(c^2 - 4bp - 4aq)^3 - 36(ce - 2ab)pq(c^2 - 4bp - 4aq) - 216c^2p^2q^2\,, \end{aligned}$$

and then by symmetry the other equation has the same invariants. The absolute invariant $I^3 \div J^2$ has therefore the same value in the two equations; that is, the equations are linearly transformable the one into the other, which is the before-mentioned theorem that the two pencils are homographic.

144. The two equations will be satisfied by $\theta = \phi$, if only $bp = aq$; that is, if $p = \frac{a}{k}$, $q = \frac{b}{k}$; putting for convenience $\frac{e}{k}$ in place of e, the equation of the curve is then

$$\xi\eta\,(a\xi + b\eta + cz) + kz^2(a\xi + b\eta + cz) = 0\,.$$

In this case the pencils of tangents are $a\xi = k\theta z$, $b\eta = k\theta z$, where θ is determined by a quartic equation, or taking the corresponding lines (which by their intersections determine the foci A, B, C, D) to be ($a\xi = k\theta_1 z$, $b\eta = k\theta_1 z$), &c., these four points lie in the line $a\xi - b\eta = 0$, which is a line through the centre of the curve, or point $\xi = 0$, $\eta = 0$: the formulæ just obtained belong therefore to the symmetrical case of the circular cubic. Passing to rectangular co-ordinates, writing $z = 1$, and taking $y = 0$ for the equation of the axis, it is easy to see that the equation may be written

$$(x^2 + y^2)(x - a) + k(x - b) = 0\,;$$

or, changing the origin and constants,

$$xy^2 + (x - a)(x - b)(x - c) = 0\,.$$

Analytical Theory for the Bicircular Quartic—Art. Nos. 145 to 149.

145. The equation for the bicircular quartic may be taken to be

$$k(\xi^2 - \alpha^2z^2)(\eta^2 - \beta^2z^2) + cz^2\xi\eta + z^3(a\xi + b\eta) + cz^4 = 0\,,$$

viz. (ξ, η, z) being any co-ordinates whatever, this is the equation of a quartic curve having a node at each of the points ($\xi = 0$, $z = 0$) and ($\eta = 0$, $z = 0$): the equations of the two tangents at the one node are $\xi - \alpha z = 0$, $\xi + \alpha z = 0$; and those of the two tangents at the other node are $\eta - \beta z = 0$, $\eta + \beta z = 0$; $\xi = 0$ is thus the harmonic of the line $z = 0$ in regard to the tangents at ($\xi = 0$, $z = 0$), and $\eta = 0$ is the harmonic of the same line $z = 0$ in regard to the tangents at $(\eta = 0, z = 0)$. If $(\xi, \eta, z = 1)$ be circular co-ordinates, then we have the general equation of the bicircular quartic having the lines $\xi + \alpha z = 0$, $\xi - \alpha z = 0$ for one pair, and the lines $\eta - \beta z = 0$, $\eta + \beta z = 0$ for the other pair of parallel asymptotes; and therefore the point $\xi = 0$, $\eta = 0$ for centre, and the lines $\beta\xi - \alpha\eta = 0$, $\beta\xi + \alpha\eta = 0$ for nodal axes. In order that the curve may be real we must have (α, β), (a, b) conjugate imaginaries, k, e, c real. The points $(\xi = 0, z = 0)$ and $(\eta = 0, z = 0)$ are as before the points I, J. If $\alpha = 0$, the node at I becomes a cusp, and so if $\beta = 0$, the node at J becomes a cusp; the form thus includes the case of a bicuspidal or Cartesian curve.

146. To find the tangents from I, writing in the equation of the curve $\xi = \theta\alpha z$, we have

$$k\boldsymbol{\alpha}^2(\theta^2 - 1)(\eta^2 - \beta^2 z^2) + e\boldsymbol{\alpha}\theta\eta z + z(a\boldsymbol{\alpha}\theta z + b\eta) + cz^2 = 0\,;$$

that is

$$\begin{aligned} &\eta^2\,.\,k\boldsymbol{\alpha}^2(\theta^2 - 1),\\ +\ &\eta z\,.\,e\boldsymbol{\alpha}\theta + b,\\ +\ &z^2\,.\,- k\boldsymbol{\alpha}^2\beta^2(\theta^2 - 1) + a\boldsymbol{\alpha}\theta + c = 0, \end{aligned}$$

and the condition of tangency is

$$4k(\theta^2 - 1)\{k\boldsymbol{\alpha}^2\beta^2(\theta^2 - 1) - a\boldsymbol{\alpha}\theta - c\} + \left(e\theta + \frac{b}{\boldsymbol{\alpha}}\right)^2 = 0\,;$$

viz., the tangents from I are $\xi = \theta\alpha z$, where θ is any root of this equation. Similarly, if we have

$$4k(\varphi^2 - 1)\{k\boldsymbol{\alpha}^2\beta^2(\varphi - 1) - b\beta\varphi - c\} + \left(e\varphi + \frac{a}{\beta}\right)^2 = 0\,,$$

the tangents from J are $\eta = \phi\beta z$, where ϕ is any root of this equation.

147. The two equations may be written

$$\left\{\begin{array}{ll} & 24k^2\boldsymbol{\alpha}^2\beta^2\,, \\ - & 6ka\boldsymbol{\alpha}\,, \\ - & 8k^2\boldsymbol{\alpha}^3\beta^2 - 4kc + e^2\,, \\ & 6a\boldsymbol{\alpha} \qquad\qquad + 3e\dfrac{b}{\boldsymbol{\alpha}}\,, \\ & 24k^2\boldsymbol{\alpha}^2\beta^2 + 24kc + 6\dfrac{b^2}{\boldsymbol{\alpha}^2} \end{array}\right\}(\theta, 1)^4 = 0,\ \left\{\begin{array}{ll} & 24k^2\boldsymbol{\alpha}^2\beta^3\,, \\ - & 6kb\beta\,, \\ - & 8k^2\boldsymbol{\alpha}^2\beta^2 - 4kc + e^3\,, \\ & 6kb\beta \qquad\qquad + 3e\dfrac{\boldsymbol{\alpha}}{\beta}\,, \\ & 24k^2\boldsymbol{\alpha}^2\beta^2 + 24kc + 6\dfrac{\boldsymbol{\alpha}^2}{\beta^2} \end{array}\right\}(\varphi, 1)^4 = 0\,,$$

which equations have the same invariants; in fact for the first equation the invariants are found to be as follows, viz., if for shortness

$$\mathrm{C} = - 8k^2\boldsymbol{\alpha}^2\beta^2 - 4kc + e^2,$$

then

$$I = 576k^4\boldsymbol{\alpha}^4\beta^4 + 576k^3c\boldsymbol{\alpha}^2\beta^2 + 144k^2(a^2\boldsymbol{\alpha}^2 + b^2\beta^2) + 72kab + 3C^2\,,$$

$$\begin{aligned} J = \mathrm{C}\{&576k^4\boldsymbol{\alpha}^4\beta^4 + 576k^3c\boldsymbol{\alpha}^2\beta^2 + 144k^2(a^2\boldsymbol{\alpha}^2 + b^2\beta^2) + 36kc\boldsymbol{\alpha}\beta - C^2\}\\ &- 864k^3eab\boldsymbol{\alpha}^2\beta^2 - 216k^2e^2(a^2\boldsymbol{\alpha}^2 + b^2\beta^2) - 216k^2a^2b^2\,, \end{aligned}$$

and then by symmetry the other equation has the same invariants. The absolute invariant $I^3 \div J^2$ has thus the same value in the two equations, that is, the equations are linearly transformable the one into the other, which is the before-mentioned theorem that the pencils are homographic.

148. The equations will be satisfied by $\theta = \phi$ if only $a\alpha = b\beta$, that is, if $a, b = m\beta, m\alpha$; or by $\theta = -\phi$ if only $a\alpha = -b\beta$, that is, if $a, b = m\beta, -m\alpha$: the equation of the curve is these two cases respectively—

$$k(\xi^2 - \alpha^2 z^2)(\eta^2 - \beta^2 z^2) + ez^2\xi\eta + mz^3(\beta\xi + \alpha\eta) + cz^4 = 0\,,$$
$$k(\xi^2 - \alpha^2 z^2)(\eta^2 - \beta^2 z^2) + ez^2\xi\eta + mz^3(\beta\xi - \alpha\eta) + cz^4 = 0\,.$$

If to fix the ideas we attend to the first case, then the equation in θ is

$$\left\{\begin{array}{l} \ 24k^2\alpha^2\beta^2, \\ -\ \ 6km\alpha\beta, \\ -\ \ 8k^2\alpha^2\beta^2 - \ 4kc + e^2, \\ \ \ 6km\alpha\beta \qquad\qquad + 3me, \\ \ 24k^2\alpha^2\beta^2 + 24kc + 6m^2 \end{array}\right\}(\theta, 1)^4 = 0\,;$$

and we may take as corresponding tangents through the two nodes respectively $\xi = \theta\alpha z$, $\eta = \theta\beta z$; the foci A, B, C, D, which are the intersections of the pairs of lines ($\xi = \theta_1\alpha z$, $\eta = \theta_1\beta z$), &c., lie, it is clear, in the line $\beta\xi - \alpha\eta = 0$, which is one of the nodal axes of the curve. Similarly, in the second case, if θ be determined by the foregoing equation, we may take as corresponding tangents through the two nodes respectively $\xi = \theta\alpha z$, $\eta = -\theta\beta z$; the foci (A, B, C, D), which are the intersections of the pairs of lines ($\xi = \theta_1\alpha z$, $\eta = -\theta_1\beta z$), &c., lie in the line $\beta\xi + \alpha\eta = 0$, which is the other of the nodal axes of the curve. In either case the foci A, B, C, D lie in a line, that is, we have the curve symmetrical; and, as we have just seen, the focal axis, or axis of symmetry, is one or other of the nodal axes.

149. In the case of the Cartesian, or when $\alpha = 0$, $\beta = 0$, viz., the equation $a\alpha = b\beta$ is satisfied identically, and this seems to show that the Cartesian is symmetrical; it is to be observed, however, that for $\alpha = 0$, $\beta = 0$ the foregoing formulæ fail, and it is proper to repeat the investigation for the special case in question. Writing $\alpha = 0$, $\beta = 0$ the equation of the curve is

$$k\xi^2\eta^2 + ez^2\xi\eta + z^3(a\xi + b\eta) + cz^4 = 0,$$

and then, taking $\xi = \theta bz$ for the equation of the tangent from I, we have

$$\begin{aligned} &\eta^2 \,.\, kb^2\theta^2 \\ &+ \eta z \,.\, b(e\theta + 1) \\ &+ z^2 \,.\, ab\theta + c = 0. \end{aligned}$$

and the condition of tangency is

$$4k\theta^2(ab\theta + c) - (e\theta + 1)^2 = 0\,;$$

viz., we have here a cubic equation. Similarly, if we have $\eta = \theta az$ for the equation of a tangent from J, then

$$4k\phi^2(ab\phi + c) - (e\phi + 1)^2 = 0.$$

Hence θ being determined by the cubic equation as above, we may take $\phi=\theta$, and consequently the equations of the corresponding tangents will be $\xi=\theta bz$, $\eta=\theta az$, viz., the foci A, B, C will be given as the intersections of the pairs of lines ($\xi=\theta_1 bz$, $\eta=\theta_1 az$), &c. The foci lie therefore in the line $a\xi-b\eta=0$; or the curve is symmetrical, the focal axis, or axis of symmetry, passing through the centre.

On the Property that the Points of Contact of the Tangents from a Pair of Concyclic Foci lie in a Circle—Art. Nos. 150 to 158.

150. We have seen that the foci form four concyclic sets (A, B, C, D), (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3), that is, A, B, C, D are in a circle. We may, if we please, say that any one focus is concyclic—viz., it lies in a circle with three other foci; but any two foci taken at random are not concyclic; it is only a pair such as (A, B) taken out of a set of four concyclic foci which are concyclic, viz., there exist two other foci lying with them in a circle. The number of such pairs is, it is clear $= 24$. Let A, B be any two concyclic foci, I say that the points of contact of the tangents AI, AJ, BI, BJ, lie in a circle.

151. Consider the case of the bi-circular quartic, and take as before ($\xi=0$, $z=0$), and ($\eta=0$, $z=0$) for the co-ordinates of the points I, J respectively. Let the two tangents from the focus A be $\xi-az=0$, $\eta-a'z=0$, say for shortness $p=0$, $p'=0$, then the equation of the curve is expressible in the form $pp'U=V^2$[*], where $U=0$, $V=0$ are each of them circles, viz., U and V are each of them quadric functions containing the terms z^2, $z\eta$, $z\xi$, and $\xi\eta$. Taking an indeterminate coefficient λ, the equation may be written

$$pp'(U+2\lambda V+\lambda^2 pp')=(V+\lambda pp')^2,$$

and then λ may be so determined that $U+2\lambda V+\lambda^2 pp'=0$, shall be a 0-circle, or pair of lines through I and J. It is easy to see that we have thus for λ a cubic equation, that is, there are three values of λ, for each of which the function $U+2\lambda V+\lambda^2 pp'$ assumes the form $(\xi-\beta z)(\eta-\beta' z)$, $=qq'$ suppose: taking any one of these, and changing the value of V so as that we may have V in place of $V+\lambda pp'$, the equation is $pp'qq'+V^2$, where $V=0$ is as before a circle, the equation shows that the points of contact of the tangents $p=0$, $p'=0$, $q=0$, $q'=0$ lie in this circle $V=0$. The circumstance that λ is determined by a cubic equation would suggest that the focus $q=0$, $q'=0$ is one of the three foci B, C, D concyclic with A; but this is the very thing which we wish to prove, and the investigation, though somewhat long, is an interesting one.

152. Starting from the form $pp'qq'=V^2$, then introducing as before an arbitrary coefficient λ, the equation may be written

$$pp'(qq'+2\lambda V+\lambda^2 pp')=(V+\lambda pp')^2,$$

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[*] This investigation is similar to that in Salmon's Higher Plane Curves, p. 196, in regard to the double tangents of a quartic curve.

and we may determine λ so that $qq' + 2\lambda V + \lambda^2 pp' = 0$ shall be a pair of lines. Writing $V = H\xi\eta - L\eta z - L'\xi z + Mz^2$, and substituting for pp' and qq' their values $(\xi - \alpha z)(\eta - \alpha' z)$ and $(\xi - \beta z)(\eta - \beta' z)$, the equation in question is

$$(1 + 2\lambda H + \lambda^2)\xi\eta - (\beta + 2\lambda L + \lambda^2\alpha)\eta z - (\beta' + 2\lambda L' + \lambda^2\alpha')\xi z + (\beta\beta' + 2\lambda M + \lambda^2\alpha\alpha')z^2 = 0,$$

and the required condition is

$$(1 + 2\lambda H + \lambda^2)(\beta\beta' + 2\lambda M + \lambda^2\alpha\alpha') = (\beta + 2\lambda L + \lambda^2\alpha)(\beta' + 2\lambda L' + \lambda^2\alpha');$$

or reducing, this is

$$\begin{aligned}&(2M + 2H\beta\beta' - 2L'\beta - 2L\beta)\\ &+ \lambda\big((\alpha - \beta)(\alpha' - \beta') + 4HM - 4LL'\big)\\ &+ \lambda^2(2M + 2H\alpha\alpha' - 2L'\alpha - 2L\alpha') = 0,\end{aligned}$$

viz., λ is determined by a quadric equation. Calling its roots λ_1, and λ_2, the foregoing equation, substituting therein successively these values, becomes $(\xi - \gamma z)(\eta - \gamma' z) = 0$, and $(\xi - \delta z)(\eta - \delta' z) = 0$ respectively, say $rr' = 0$ and $ss' = 0$.

153. We have to show that the four foci ($p = 0$, $p' = 0$), ($q = 0$, $q' = 0$), ($r = 0$, $r' = 0$), ($s = 0$, $s' = 0$) are a set of concyclic foci; that is, that the lines $p = 0$, $q = 0$, $r = 0$, $s = 0$ correspond homographically to the lines $p' = 0$, $q' = 0$, $r' = 0$, $s' = 0$; or, what is the same thing, that we have

$$\begin{vmatrix} 1, \alpha, \alpha', \alpha\alpha' \\ 1, \beta, \beta', \beta\beta \\ 1, \gamma, \gamma', \gamma\gamma' \\ 1, \delta, \delta', \delta\delta' \end{vmatrix} = 0,$$

or, as it will be convenient to write this equation,

$$\frac{\alpha - \beta}{\alpha' - \beta'}\frac{\gamma - \delta}{\gamma' - \delta} = \frac{\alpha - \delta}{\alpha' - \delta'}\frac{\beta - \gamma}{\beta' - \gamma'}.$$

154. We have

$$\gamma = \frac{\beta + 2\lambda_1 L + \lambda_1^2\alpha}{1 + 2H\lambda_2 + \lambda_1^2}, \quad \gamma' = \frac{\beta' + 2\lambda_1 L' + \lambda'_1\alpha'}{1 + 2H\lambda_1 + \lambda_1^2},$$

$$\delta = \frac{\beta + 2\lambda_2 L + \lambda_2^2\alpha}{1 + 2H\lambda_2 + \lambda_2^2}, \quad \delta' = \frac{\beta' + 2\lambda_2 L' + \lambda_2^2\alpha'}{1 + 2H\lambda_2 + \lambda_2^2}.$$

The expressions of $\alpha - \delta$, &c., are severally fractions, the denominators of which disappear from the equation; the numerators are

$$\begin{aligned}\text{for } \alpha - \delta, &= \alpha(1 + 2\lambda_2 H + \lambda_2^2) - (\beta + 2\lambda_2 L + \alpha\lambda_2^2),\\ &= \alpha - \beta + 2\lambda_2(\alpha H - L);\\ \text{for } \beta - \gamma, &= \beta(1 + 2\lambda_1 H + \lambda_1^2) - (\beta + 2\lambda_1 L + \alpha\lambda_1^2),\\ &= \lambda_1\{2(\beta H - L)(\alpha - \beta)\};\\ \text{for } \gamma - \delta, &= (\beta + 2L\lambda_1 + \alpha\lambda_1^2)(1 + 2H\lambda_2 + \lambda_2^2)\\ &\quad - (\beta + 2L\lambda_2 + \alpha\lambda_2^2)(1 + 2H\lambda_1 + \lambda_1^2),\\ &= (\alpha' - \beta')\{2H^2\alpha\beta - 2HL(\alpha + \beta) + 2L^2 + \tfrac{1}{2}(\alpha - \beta)^2\}\end{aligned}$$

and it hence easily appears that the equation to be verified is

$$\frac{2H^2\alpha\beta - 2HL(\alpha + \beta) + 2L^2 + \tfrac{1}{2}(\alpha - \beta)^2}{2H^2\alpha'\beta' - 2HL'(\alpha' + \beta')\,2L'^2 + \tfrac{1}{2}(\alpha' - \beta')^2} = \frac{\alpha - \beta + 2(\alpha H - L)\lambda_2}{\alpha' - \beta' + 2(\alpha' H - L')\lambda_2} \cdot \frac{2(\beta H - L) - (\alpha - \beta)\lambda_1}{2(\beta' H - L') - (\alpha' - \beta')\lambda_2}.$$

155. This is

$$\frac{B - C}{B' - C'} = \frac{A + B\lambda_1 + C\lambda_2 + D\lambda_1\lambda_2}{A' + B'\lambda_1 + C'\lambda_2 + D'\lambda_1\lambda_2},$$

if for shortness

$$\begin{array}{ll} A = 2(\alpha - \beta)(\beta H - L), & A' = 2(\alpha' - \beta')(\beta' H - L')\;, \\ B = -(\alpha - \beta)^2\;, & B' = -(\alpha' - \beta')^2\;, \\ C = 4(\alpha H - L)(\beta - L), & C' = 4(\alpha' H - L')(\alpha' H - L'), \\ D = -2(\alpha - \beta)(\alpha H - L), & D' = -2(\alpha' - \beta')(\alpha' H - L'), \end{array}$$

and the equation then is

$$AB' - A'B + CA' - C'A - (\lambda_1 + \lambda_2)(B'C - B'C) + \lambda_1\lambda_2\big(CD' - C'D - (BD' - B'D)\big).$$

156. Calculating $AB' - A'B$, $CA' - C'A$, $CD' - C'D$, $BD' - B'D$, these are at once seen to divide by $\{(\alpha\beta' - \alpha'\beta)H + L(\alpha' - \beta') - L'(\alpha - \beta')\}$; we have, moreover,

$$\begin{aligned} BC' - B'C &= -4(\alpha - \beta)^2(\alpha' H - L')(\beta' H - L') + 4(\alpha' - \beta')^2(\alpha H - L)(\beta H - L) \\ &= -\{(\alpha\alpha' - \beta\beta')H - L(\alpha' - \beta') - L'(\alpha - \beta)\}\{(\alpha\beta' - \alpha'\beta)H + L(\alpha' - \beta') - L'(\alpha - \beta)\}, \end{aligned}$$

viz., this also contains the same factor; and omitting it, the equation is found to be

$$\begin{aligned} &\{(\alpha - \beta)(\alpha' - \beta') - 4(\beta H - L)(\beta' H - L')\} \\ &-2\{(\alpha\alpha' - \beta\beta')H - L(\alpha' - \beta') - L'(\alpha - \beta)\}(\lambda_1 + \lambda_2) \\ &+\{-(\alpha - \beta)(\alpha' - \beta') + 4(\alpha H - L)(\alpha' H - L')\}\lambda_1\lambda_2 = 0; \end{aligned}$$

viz., substituting for $\lambda_1 + \lambda_2$ and $\lambda_1\lambda_2$ their values, this is

$$\begin{aligned} &\{(\alpha - \beta)(\alpha' - \beta') - 4(\beta H - L)(\beta' H - L')\}(M + H\alpha\alpha' - L\alpha' - L'\alpha) \\ &-\{(\alpha\alpha' - \beta\beta')H - L(\alpha' - \beta')\}\{(\alpha - \beta)(\alpha' - \beta') + 4HM - 4LL'\} \\ &+\{-(\alpha - \beta)(\alpha' - \beta') + 4(\alpha H - L)(\alpha' H - L')\}\{M + H\beta\beta' - L\beta' - L'\beta\} = 0, \end{aligned}$$

which should be identically true. Multiplying by H, and writing in the form

$$\begin{aligned} &\{(\alpha - \beta)(\alpha' - \beta') - 4(\beta H - L)(\beta' H - L')\}\big(HM - LL' + (\alpha H - L)(\alpha' H - L')\big) \\ &-\{(\alpha H - L)(\alpha' H - L') - (\beta H - L)(\beta' H - L')\}\big((\alpha - \beta)(\alpha' - \beta) + 4(HM - LL')\big) \\ &+\{-(\alpha - \beta)(\alpha' - \beta') + 4(\alpha H - L)(\alpha' H - L')\}\big(HM - LL' + (\beta H - L)(\beta' H - L')\big) = 0, \end{aligned}$$

we at once see that this is so, and the theorem is thus proved, viz., that the equation being $pp'qq' = V^2$, the foci $(p = 0, p' = 0)$ and $(q = 0, q' = 0)$ are concyclic.

157. By what precedes, λ being a root of the foregoing quadric equation, we may write

$$qq' + 2\lambda V + \lambda^2 pp' = K^2 rr',$$

where the focus $r = 0$, $r' = 0$ is concyclic with the other two foci; but from the equation of the curve $V = \sqrt{pp'qq'}$, that is we have

$$qq' + 2\lambda\sqrt{pp'qq'} + \lambda^2 pp' = Krr',$$

or, what is the same thing,

$$\lambda\sqrt{pp'} + \sqrt{qq'} + K\sqrt{rr'} = 0,$$

viz., this is a form of the equation of the curve; substituting for p, p', q, q', r, r' their values, writing also

$$\begin{aligned}\mathsf{A} &= (\xi - \alpha z)(\eta - \alpha' z),\\ \mathsf{B} &= (\xi - \beta z)(\eta - \beta' z),\\ \mathsf{C} &= (\xi - \gamma z)(\eta - \gamma' z),\end{aligned}$$

and changing the constants λ, K (viz. $\lambda : 1 : K = \sqrt{l} : \sqrt{m} : \sqrt{n}$) the equation is

$$\sqrt{l\mathsf{A}} + \sqrt{m\mathsf{B}} + \sqrt{n\mathsf{C}} = 0,$$

viz., we have the theorem that for a bicircular quartic if $(\xi - \alpha z = 0, \eta - \alpha' z = 0)$, $(\xi - \beta z = 0, \eta - \beta' z = 0$, $(\xi - \gamma z = 0)$, $\eta - \gamma' z = 0)$ be any three concyclic foci, then the equation is as just mentioned; that is, the curve is a trizomal curve, the zomals being the three given foci regarded as 0-circles. The same theorem holds in regard to the circular cubic, and a similar demonstration would apply to this case.

158. It may be noticed that we might, without proving as above that the two foci $(p = 0, p' = 0)$, $(q = 0, q' = 0)$ were concyclic, have passed at once from the form $pp'qq' = V^2$, to the form $\sqrt{\lambda pp'} + \sqrt{qq'} + K\sqrt{rr'} = 0$ (or $\sqrt{l\mathsf{A}} = \sqrt{m\mathsf{B}} = \sqrt{n\mathsf{C}} = 0$), and then by the application of the theorem of the variable zomal (thereby establishing the existence of a fourth focus concyclic with the three) have shown that the original two foci were concyclic. But it seemed the more orderly course to effect the demonstration without the aid furnished by the reduction of the equation to the trizomal form.

PART IV. (NOS. 159 TO 206).—ON TRIZOMAL AND TETRAZOMAL CURVES WHERE THE ZOMALS ARE CIRCLES.

The Trizomal Curve—The Tangents at I, J, &c.—Art. Nos. 159 to 165.

159. I consider the trizomal

$$\sqrt{l\mathsf{A}^\circ} + \sqrt{m\mathsf{B}^\circ} + \sqrt{n\mathsf{C}^\circ} = 0,$$

where A, B, C being the centres of three given circles, A°, &c. denote as before, viz., in rectangular and in circular co-ordinates respectively, we have

$$\begin{aligned}\mathsf{A}^\circ &= (x - az)^2 + (y - a'z)^2 - a''^2z^2, = (\xi - \alpha z)(\eta - \alpha' z) - a''^2z^2,\\ \mathsf{B}^\circ &= (x - bz)^2 + (y - b'z)^2 - b''^2z^2, = (\xi - \beta z)(\eta - \beta' z) - b''^2z^2,\\ \mathsf{C}^\circ &= (x - cz)^2 + (y - c'z)^2 - c''^2z^2, = (\xi - \gamma z)(\eta - \gamma' z) - c''^2z^2.\end{aligned}$$

By what precedes, the curve is of the order $= 4$, touching each of the given circles twice, and having a double point, or node, at each of the points I, J; that is, it is a bicircular quartic: but if for any determinate values of the radicals $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$, we have

$$\sqrt{l} + \sqrt{m} + \sqrt{n} = 0,$$

then there is a branch

$$\sqrt{l\mathrm{A}^\circ} + \sqrt{m\mathrm{B}^\circ} + \sqrt{n\mathrm{C}^\circ} = 0,$$

containing ($z = 0$) the line infinity; and the order is here $= 3$: viz., the curve here passes through each of the points I, J and through another point at infinity (that is, there is an asymptote), and is thus a circular cubic.

160. I commence by investigating the equations of the nodal tangents at the points I, J respectively; using for this purpose the circular co-ordinates ($\xi, \eta, z = 1$), it is to be observed that, in the rationalised equation, for finding the tangents at ($\xi = 0$, $z = 0$) we have only to attend to the terms of the second order in (ξ, z), and similarly for finding the tangents at ($\eta = 0$, $z = 0$) we have only to attend to the terms of the second order in (η, z). But it is easy to see that on any term involving a'', b'', or c'' will be of the third order at least in (ξ, z), and similarly of the third order at least in (η, z); hence for finding the tangents we may reject the terms in question, or, what is the same thing, we may write a'', b'', c'' each $= 0$, thus reducing the three circles to their respective centres. The equation thus becomes

$$\sqrt{l(\xi - \alpha z)(\eta - \alpha' z)} + \sqrt{m(\xi - \beta z)(\eta - \beta' z)} + \sqrt{n(\xi - \gamma z)(\eta - \gamma' z)} = 0.$$

For finding the tangents at ($\xi = 0$, $z = 0$) we have in the rationalised equation to attend only to the terms of the second order in (ξ, z); and it is easy to see that any term involving α', β', γ' will be of the third order at least in (ξ, z), that is, we may reduce α', β', γ' each to zero; the irrational equation then becomes divisible by $\sqrt{\eta}$, and throwing out this factor, it is

$$\sqrt{l(\xi - \alpha z)} + \sqrt{m(\xi - \beta z)} + \sqrt{n(\xi - \gamma z)} = 0,$$

viz., this equation which evidently belongs to a pair of lines through the point ($\xi = 0$, $z = 0$) gives the tangents at the point in question; and similarly the tangents at the point ($\eta = 0$, $z = 0$) are given by the equation

$$\sqrt{l(\eta - \alpha' z)} + \sqrt{m(\eta - \beta' z)} + \sqrt{n(\eta - \gamma' z)} = 0.$$

161. To complete the solution, attending to the tangents at ($\xi = 0$, $z = 0$), and putting for shortness

$$\begin{aligned}
\lambda &= l - m - n,\\
\mu &= -l + m - n,\\
\nu &= -l - m + n,\\
\Delta &= l^2 + m^2 + n^2 - 2mn - 2nl - 2lm,
\end{aligned}$$

the rationalised equation is easily found to be

$$\begin{aligned}&\xi^2 \,.\, \Delta\\ &- 2\xi z(l\lambda\alpha + m\mu\beta + n\nu\gamma)\\ &+ \; z^2(l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta) = 0 \,.\end{aligned}$$

And it is to be noticed that in the case of the circular cubic or when $\sqrt{l} + \sqrt{m} + \sqrt{n} = 0$, then $\Delta = 0$, so that the equation contains the factor z, and throwing this out, the equation gives a single line, which is in fact the tangent of the circular cubic.

162. Returning to the bicircular quartic, we may seek for the condition in order that the node may be a cusp: the required condition is obviously

$$\Delta(l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta) - (l\lambda\alpha + m\mu\beta + n\nu\gamma)^2 = 0,$$

or observing that

$$\begin{aligned}\Delta - \lambda^2 &= -\, 4mn \,, \text{\&c.}\\ \Delta + \mu\nu &= -\, 2l\lambda \,, \text{\&c.}\end{aligned}$$

this is

$$l\alpha^2 + m\beta^2 + n\gamma^2 + \lambda\beta\gamma + \mu\gamma\alpha + \nu\alpha\beta = 0 \,,$$

or substituting for λ, μ, ν, their values, it is

$$l(\alpha - \beta)(\alpha - \gamma) + m(\beta - \gamma)(\beta - \alpha) + n(\gamma - \alpha)(\gamma - \beta) = 0 \,,$$

or as it is more simply written

$$\frac{l}{\beta - \gamma} + \frac{m}{\gamma - \alpha} + \frac{n}{\alpha - \beta} = 0 \,.$$

163. If the node at ($\eta = 0$, $z = 0$) be also a cusp, then we have in like manner

$$\frac{l}{\beta' - \gamma'} + \frac{m}{\gamma' - \alpha'} + \frac{n}{\alpha' - \beta'} = 0 \,.$$

Now observing that

$$\begin{aligned}(\gamma - \alpha)(\alpha' - \beta') - (\gamma' - \alpha')(\alpha - \beta) \,, &= \begin{vmatrix} \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \end{vmatrix}\\ = (\alpha - \beta)(\beta' - \gamma') - (\alpha' - \beta')(\beta - \gamma) \,, &\\ = (\beta - \gamma)(\gamma' - \alpha') - (\beta' - \gamma')(\gamma - \alpha) \,, &\end{aligned}$$

$= \Omega$ suppose: the two equations give

$$l : m : n = \Omega(\beta - \gamma)(\beta' - \gamma') : \Omega(\gamma - \alpha)(\gamma' - \alpha') : \Omega(\alpha - \beta)(\alpha' - \beta') \,;$$

or if Ω is not $= 0$, then

$$l : m : n = (\beta - \gamma)(\beta' - \gamma') : (\gamma - \alpha)(\gamma' - \alpha') : (\alpha - \beta)(\alpha' - \beta') \,.$$

164. If

$$\Omega = \begin{vmatrix} \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \end{vmatrix} , = 0 \,,$$

or, what is the same thing, if

$$\begin{vmatrix} a, & a', & 1 \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix} = 0,$$

the centres A, B, C are in a line; taking it as the axis of x, we have $\alpha=\alpha'=a$, $\beta=\beta'=b$, $\gamma=\gamma'=c$; and the conditions for the cusps at I, J respectively reduce themselves to the single condition

$$\frac{l}{b-c}+\frac{m}{c-a}+\frac{n}{a-b}=0,$$

so that this condition being satisfied, the curve

$$\sqrt{l\{(x-az)^2+y^2-a''^2z^2\}}+\sqrt{m\{(x-bz)^2+y^2-b''^2z^2\}}+\sqrt{n(x-cz)^2+y^2-c''^2z^2\}}=0$$

is a Cartesian; viz., given any three circles with their centres on a line, there are a singly infinite series of Cartesians, each touched by the three circles respectively; the line of centres is the axis of the curve, but the centres A, B, C are not the foci, except in the case $a''=0$, $b''=0$, $c''=0$, where the circles vanish. The condition for l, m, n is satisfied if $l:m:n=(b-c)^2:(c-a)^2:(a-b)^2$; these values writing $\sqrt{l}:\sqrt{m}:\sqrt{n}=b-c:c-a:a-b$, give not only $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, but also $a\sqrt{l}+b\sqrt{m}+c\sqrt{n}=0$; these are the conditions for a branch containing ($z^2=0$) the line infinity twice; the equation

$$(b-c)\sqrt{(x-az)^2+y^2-a''^2z^2}+(c-a)\sqrt{(x-bz)^2+y^2-b''^2z^2}+(a-b)\sqrt{(x-cz)^2+y^2-c''^2z^2}=0,$$

is thus that of a conic, and if $a''=0$, $b''=0$, $c''=0$, then the curve reduces itself to $y^2=0$, the axis twice.

165. If Ω is not $=0$, then we have

$$l:m:n=(\beta-\gamma)(\beta'-\gamma'):(\gamma-\alpha)(\gamma'-\alpha'):(\alpha-\beta)(\alpha'-\beta'),$$

viz., l, m, n are as the squared distances $\overline{BC}^2$, $\overline{CA}^2$, $\overline{AB}^2$, say as $f^2:g^2:h^2$; or when the centres of the given circles A, B, C are not in a line, then f, g, h being the distances BC, CA, AB of these centres from each other, we have, touching each of the given circles twice, the *single* Cartesian

$$f\sqrt{\mathrm{A}^\circ}+g\sqrt{\mathrm{B}^\circ}+h\sqrt{\mathrm{C}^\circ}=0,$$

which, in the particular case where the radii a'', b'', c'' are each $=0$, becomes

$$f\sqrt{\mathrm{A}}+g\sqrt{\mathrm{B}}+h\sqrt{\mathrm{C}}=0,$$

viz., this is the circle through the points A, B, C, say the circle ABC, twice.

Investigation of the Foci of a Conic represented by an Equation in Areal Co-ordinates— Art. Nos. 166 to 169.

166. I premise as follows: Let A, B, C be any given points, and in regard to the triangle ABC let the *areal* co-ordinates of a current point P be u, v, w; that is, writing PBC, &c., for the areas of these triangles, take the co-ordinates to be

$$u : v : w = PBC : PCA : PAB\,,$$

or, what is the same thing in the rectangular co-ordinates $(x, y, z = 1)$, if

$$(a, a', 1),\ (b, b', 1),\ (c, c', 1)\,,$$

be the co-ordinates of A, B, C respectively, take

$$u : v : w = \begin{vmatrix} x, & y, & z \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ c, & c', & 1 \\ a, & a', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ a, & a', & 1 \\ b, & b', & 1 \end{vmatrix}$$

or in the circular co-ordinates $(\xi, \eta, z = 1)$, if $(\alpha, \alpha', 1), (\beta, \beta', 1), (\gamma, \gamma', 1)$ be the co-ordinates of the three points respectively, then

$$u : v : w = \begin{vmatrix} \xi, & \eta, & z \\ \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \end{vmatrix} : \begin{vmatrix} \xi, & \eta, & z \\ \gamma, & \gamma', & 1 \\ \alpha, & \alpha', & 1 \end{vmatrix} : \begin{vmatrix} \xi, & \eta, & z \\ \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \end{vmatrix}$$

167. For the point I we have $(\xi, \eta, z) = (0, 1, 0)$, and hence if its areal co-ordinates be (u_0, v_0, w_0), we have

$$u_0 : v_0 : w_0 = \beta - \gamma : \gamma - \alpha : \alpha - \beta\,,$$

and hence also, (u, v, w) referring to the current point P, we find

$$\begin{aligned} v_0 w - w_0 v = (\gamma - \alpha)[(\alpha' - \beta')(\xi - \alpha z) - (\alpha - \beta)(\eta - \alpha' z)] \\ - (\alpha - \beta)[(\gamma' - \alpha')(\xi - \alpha z) - (\gamma - \alpha)(\eta - \alpha' z)] = \Omega(\xi - \alpha z)\,, \end{aligned}$$

if

$$\Omega = (\gamma - \alpha)(\alpha' - \beta') - (\alpha - \beta)(\gamma' - \alpha') = \begin{vmatrix} \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \end{vmatrix};$$

whence

$$v_0 w - w_0 v : w_0 u - w u_0 : u_0 v - u v_0 = \xi - \alpha z : \xi - \beta z : \xi - \gamma z\,,$$

and in precisely the same manner, if $u_0{}', v_0{}', w_0{}'$ refer to the point J, then

$$u'_0 : v'_0 : w'_0 = \beta' - \gamma' : \gamma' - \alpha' : \alpha' - \beta'\,,$$

and

$$v'_0 w - w'_0 v : w'_0 u - w u'_0 : u'_0 v - u v'_0 = \eta - \alpha' z : \eta - \beta' z : \eta - \gamma' z\,.$$

168. Consider the conic

$$(a, b, c, f, g, h)(u, v, w)^2 = 0\,,$$

where u, v, w are any trilinear co-ordinates whatever; and take the inverse co-efficients to be (A, B, C, F, G, H) $(A = bc - f^2$, &c.), then for any given point the co-ordinates of which are (u_0, v_0, w_0), the equation of the tangents from this point to the conic is, as is well known,

$$(A, B, C, F, G, H)(v_0 w - w_0 v, w_0 u - u_0 w, u_0 v - v_0 u)^2 = 0\ ;$$

consequently for the conic

$$(a, b, c, f, g, h)(u, v, w)^2 = 0\ ,$$

where (u, v, w) are areal co-ordinates referring, as above, to any three given points A, B, C, the equation of the pair of tangents from the point I to the conic is

$$(A, B, C, F, G, H)(\xi - \alpha z, \xi - \beta z, \xi - \gamma z)^2 = 0\ ,$$

and that of the pair of tangents from J is

$$(A, B, C, F, G, H)(\eta - \alpha' z, \eta - \beta' z, \eta - \gamma' z)^2 = 0\ ,$$

these two line-pairs intersecting, of course, in the foci of the conic.

169. In particular, if the conic is a conic passing through the points A, B, C, then taking its equation to be

$$lvw + mwu + nuv = 0\ ,$$

the inverse co-efficients are as $(l^2, m^2, n^2, -2mn, -2nl, -2lm)$, and we have for the equations of the two line-pairs

$$\sqrt{l(\xi - \alpha z)} + \sqrt{m(\xi - \beta z)} + \sqrt{n(\xi - \gamma z)} = 0\ ,$$
$$\sqrt{l(\eta - \alpha' z)} + \sqrt{m(\eta - \beta' z)} + \sqrt{n(\eta - \gamma' z)} = 0\ .$$

The Theorem of the Variable Zomal—Art. No. 170.

170. Consider the four circles

$$\mathbf{A}^\circ = 0, \mathbf{B}^\circ = 0, \mathbf{C}^\circ = 0, \mathbf{D}^\circ = 0\ (\mathbf{A}^\circ = (x - az)^2 + (y - a'z)^2 - a''^2 z^2, \text{\&c.}),$$

which have a common orthotomic circle; so that as before

$$\mathrm{a}\mathbf{A}^\circ + \mathrm{b}\mathbf{B}^\circ + \mathrm{c}\mathbf{C}^\circ + \mathrm{d}\mathbf{D}^\circ = 0\ ,$$

where

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = BCD : -CDA : DAB : -ABC\ .$$

I consider the first three circles as given, and the fourth circle as a variable circle cutting at right angles the orthotomic circle of the three given circles; this being so, attending only to the ratios $\mathrm{a} : \mathrm{b} : \mathrm{c}$, we may write

$$\mathrm{a} : \mathrm{b} : \mathrm{c} = DBC : DCA : DAB\ ,$$

that is, $(\mathrm{a}, \mathrm{b}, \mathrm{c})$ are proportional to the areal co-ordinates of the centre of the variable circle in regard to the triangle ABC.

171. Suppose that the centre of the variable circle is situate on a given conic, then expressing the equation of this conic in areal co-ordinates in regard to the triangle *ABC*, we have between (a, b, c) the equation obtained by substituting these values for the co-ordinates in the equation of the conic; that is, the equation of the variable circle is

$$\mathrm{a}\mathbf{A}^\circ + \mathrm{b}\mathbf{B}^\circ + \mathrm{c}\mathbf{C}^\circ = 0\,,$$

where (a, b, c) are connected by an equation,

$$(a, b, c, f, g, h)(\mathrm{a}, \mathrm{b}, \mathrm{c})^2 = 0\,.$$

Hence (A, B, C, F, G, H) being the inverse co-efficients, the equation of the envelope of the variable circle is

$$(A, B, C, F, G, H)(\mathbf{A}^\circ, \mathbf{B}^\circ, \mathbf{C}^\circ)^2 = 0\,,$$

and, in particular, if the conic be a conic passing through the points A, B, C, and such that its equation in the areal co-ordinates (u, v, w) in regard to the triangle ABC is

$$lvw + mwu + nuv = 0\,,$$

then the equation of the envelope is

$$(l^2, m^2, n^2, -mn, -nl, -lm)(\mathbf{A}^\circ, \mathbf{B}^\circ, \mathbf{C}^\circ)^2 = 0\,;$$

that is, it is

$$(1, 1, 1, -1, -1, -1)(l\mathbf{A}^\circ, m\mathbf{B}^\circ, n\mathbf{C}^\circ)^2 = 0\,,$$

or, what is the same thing, it is

$$\sqrt{l\mathbf{A}^\circ} + \sqrt{m\mathbf{B}^\circ} + \sqrt{n\mathbf{C}^\circ} = 0\,.$$

172. It has been seen that the equations of the nodal tangents at the points I, J respectively are respectively

$$\sqrt{l(\xi - \alpha z)} + \sqrt{m(\xi - \beta z)} + \sqrt{n(\xi - \gamma z)} = 0\,,$$
$$\sqrt{l(\eta - \alpha' z)} + \sqrt{m(\eta - \beta' z)} + \sqrt{n(\eta - \gamma' z)} = 0\,,$$

and that these are the equations of the tangents to the conic $lvw + mwu + nuv = 0$ from the points I, J respectively. We have thus Casey's theorem for the generation of the bi-circular quartic as follows:—The envelope of a variable circle which cuts at right angles the orthotomic circle of three given circles $\mathbf{A}^\circ = 0$, $\mathbf{B}^\circ = 0$, $\mathbf{C}^\circ = 0$, and has its centre on the conic $lvw + mwu + nuv = 0$ which passes through the centres of the three given circles is the bicircular quartic, or trizomal

$$\sqrt{l\mathbf{A}^\circ} + \sqrt{m\mathbf{B}^\circ} + \sqrt{n\mathbf{C}^\circ} = 0,$$

which has its nodo-foci coincident with the foci of the conic.

173. To complete the analytical theory, it is proper to express the equation of

the orthotomic circle by means of the areal co-ordinates (u, v, w). Writing for shortness $a^2 + a'^2 - a''^2 = \grave{a}$, &c., and therefore

$$\mathsf{A}^\circ = x^2 + y^2 - 2axz - 2a'yz - a^{\backprime}z^2, \text{ \&c.,}$$

then if as before

$$u : v : w = \begin{vmatrix} x, & y, & z \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ c, & c', & 1 \\ a, & a', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ a, & a', & 1 \\ b, & b', & 1 \end{vmatrix},$$

and therefore

$$x : y : z = au + bv + cw : a'u + b'v + c'w : u + v + w,$$

the equation of the orthotomic circle is

$$\begin{vmatrix} x - az, & y - a'z, & ax + a'y - a^{\backprime}z \\ x - bz, & y - b'z, & bx + b'y - b^{\backprime}z \\ x - cz, & x - c'z, & cx + c'y - c^{\backprime}z \end{vmatrix} = 0,$$

viz., throwing out the factor z, this is

$$u(ax + a'y - a^{\backprime}z) + v(bx + b'y - b^{\backprime}z) + w(cx + c'y - c^{\backprime}z) = 0,$$

or what is the same thing, it is

$$(au + bv + cw)\,x + (a'u + b'v + c'w)\,y - (a^{\backprime}u + b^{\backprime}v + c^{\backprime}w)z = 0,$$

viz.,it is

$$(au + bv + cw)^2 + (a'u + b'v + c'w)^2 - (a^{\backprime}u + b^{\backprime}v + c^{\backprime}w)\,(u + v + w) = 0,$$

that is, substituting for $a^{\backprime}, b^{\backprime}, c^{\backprime}$ their values, it is

$$\begin{aligned} & a''^2u^2 + b''^2v^2 + c''^2w^2 \\ & + \big(b''^2 + c''^2 - (b - c)^2 - (b' - c')^2\big)\,vw \\ & + \big(c''^2 + a''^2 - (c - a)^2 - (c' - a')^2\big)\,wu \\ & + \big(a''^2 + b''^2 - (a - b)^2 - (a' - b')^2\big)\,uv = 0, \end{aligned}$$

and it may be observed that using for a moment α, β, γ to denote the angles at which the three circles taken in pairs respectively intersect, then we have $2\,b''\,c''\cos\alpha = b''^2 + c''^2 - (b - c)^2 - (b' - c')^2$, &c., and the equation of the orthotomic circle thus is

$$(1, 1, 1, \cos\alpha, \cos\beta, \cos\gamma)\,(a''u, b''v, c''w)^2 = 0.$$

174. We have in the foregoing enunciation of the theorem made use of the three given circles A, B, C, but it is clear that these are in fact *any* three circles in the series of the variable circle, and that the theorem may be otherwise stated thus:—

The envelope of a variable circle which has its centre in a given conic, and cuts at right angles a given circle, is a bi-circular quartic, such that its nodofoci are the foci of the conic.

Properties depending on the relation between the Conic and Circle—
Art. Nos. 175 to 177.

175. I refer to the conic of the theorem simply as the conic, and to the fixed circle simply as the circle, or when any ambiguity might otherwise arise, then as the orthotomic circle. This being so, I consider the effect in regard to the trizomal curve, of the various special relations which may exist between the circle and the conic.

If the conic touch the circle, the curve has a node at the point of contact.

If the conic has with the circle a contact of the second order, the curve has a cusp at the point of contact.

If the centre of the circle lie on an axis of the conic, then the four intersections lie in pairs symmetrically in regard to this axis, or the curve has this axis as an axis of symmetry.

If the conic has double contact with the circle (this implies that the centre of the circle is situate on an axis of the conic) the curve has a node at each of the points of contact, viz., it breaks up into two circles intersecting in these two points. The centres of the two circles respectively are the two foci of the conic, which foci lie on the axis in question. Observe that in the general case there are at each of the circular points at infinity two tangents, without any correspondence of the tangents of the one pair singly to those of the other pair, and there are thus four intersections, the four foci of the conic; in the present case, where the curve is a pair of circles, the two tangents to the same circle correspond to each other, and intersect in the two foci on the axis in question. The other two foci, or anti-points of these, are each of them the intersection of a tangent of the one circle by a tangent of the other circle.

If the conic has with the circle a contact of the third order (this implies that the circle is a circle of maximum or minimum curvature, at the extremity of an axis of the conic), then the curve has at this point a tacnode, viz., it breaks up into two circles touching each other and the conic at the point in question, and having their centres at the two foci situate on that axis of the conic respectively.

176. If the conic is a parabola, then the curve is a circular cubic having the four intersections of the parabola and circle for a set of concyclic foci, and having the focus of the parabola for *centre*. The like particular cases arise, viz.,

If the circle touch the parabola, the curve has a node at the point of contact.

If the circle has, with the parabola, a contact of the second order, the curve has a cusp at the point of contact.

If the centre of the circle is situate on the axis of the parabola, then the four intersections are situate in pairs symmetrically in regard to this axis, and the curve has this axis for an axis of symmetry.

If the circle has double contact with the parabola (which, of course, implies that the centre lies on the axis), then the curve has a node at each of the points of contact, viz., the curve breaks up into a line and circle intersecting at the two points of contact, and the circle has its centre at the focus of the parabola.

If the circle has with the parabola a contact of the third order (this implies that the circle is the circle of maximum curvature, touching the parabola at its vertex), then the curve has a tacnode, viz., it breaks up into a line and circle touching each other and the parabola at the vertex, that is, the line is the tangent to the parabola at its vertex, and the circle is the circle having the focus of the parabola for its centre, and passing through the vertex, or what is the same thing, having its radius $= \frac{1}{2}$ of the semi-latus rectum of the parabola.

177. If the conic be a circle, then the curve is a bi-circular quartic such that its four nodo-foci coincide together at the centre of the circle; viz., the curve is a cartesian having the centre of the conic for its cuspo-focus, that is, for the intersection of the cuspidal tangents of the cartesian. The intersections of the conic with the other circle, or say with the orthotomic circle, are a pair of non-axial foci of the cartesian; viz., the anti-points of these are two of the axial foci. The third axial focus is the centre of the orthotomic circle.

Case of Double Contact, Casey's Equation in the Problem of Tactions—Art. No. 178.

178. In the case where the conic has double contact with the orthotomic circle, then (as we have seen) the envelope of the variable circle is a pair of circles, each touching the variable circle; or, if we start with three given circles and a conic through their centres, then the envelope is a pair of circles, each of them touching each of the three given circles; that is, we have a solution of the problem of tactions. Multiplying by 2, the equation found *ante*, No. 173, for the variable circle, and then for the moment representing it by $(\mathrm{a, b, c, f, g, h})(u, v, w)^2 = 0$; then attributing any signs at pleasure to the radicals $\sqrt{\mathrm{a}}$, $\sqrt{\mathrm{b}}$, $\sqrt{\mathrm{c}}$, the equation of a conic through the centres of the given circles, and having double contact with the orthotomic circle, will be

$$(\mathrm{a, b, c, f, g, h})(u, v, w)^2 - (u\sqrt{\mathrm{a}} + v\sqrt{\mathrm{b}} + w\sqrt{\mathrm{c}})^2 = 0\,,$$

viz., representing this equation as before by

$$lvw + mwu + nuv = 0\,,$$

we have

$$l : m : n = \mathrm{f} - \sqrt{\mathrm{bc}} : \mathrm{g} - \sqrt{\mathrm{ca}} : \mathrm{h} - \sqrt{\mathrm{ab}}\,,$$

that is, substituting for a, b, c, f, g, h their values, and taking, for instance, a, b, c $= a''\sqrt{2}$, $b''\sqrt{2}$, $c''\sqrt{2}$, we find

$$\begin{aligned} l : m : n = {} & (b'' - c'')^2 - (b - c)^2 - (b' - c')^2 \\ & : (c'' - a'')^2 - (c - a)^2 - (c' - a')^2 \\ & : (a'' - b'')^2 - (a - b)^2 - (a' - b')^2, \end{aligned}$$

that is, l, m, n are as the squares of the tangential distances (direct) of the three circles taken in pairs, and this being so, the equation of a pair of circles touching each of the three given circles is $\sqrt{l\mathbf{A}^\circ} + \sqrt{m\mathbf{B}^\circ} + \sqrt{n\mathbf{C}^\circ} = 0$. It is clear that, instead of taking the three direct tangential distances, we may take one direct tangential distance and two inverse tangential distances, viz., the tangential distances corresponding to any three centres of similitude which lie in a line; we have thus in all the equations of four pairs of circles, viz., of the eight circles which touch the three given circles. This is Casey's theorem in the problem of tactions.

The Intersections of the Conic and Orthotomic Circle are a set of four Concyclic Foci—Art. No. 179.

179. The conic of centres intersects the orthotomic circle in four points, and for each of these the radius of the variable circle is $= 0$, that is, the points in question are a set of four concyclic foci (A, B, C, D) of the curve. Regarding the foci as given, the circle which contains them is of course the orthotomic circle; and there are a singly infinite series of curves, viz., these correspond to the singly infinite series of conics which can be drawn through the given foci. As for a given curve there are four sets of concyclic foci, there are four different constructions for the curve, viz., the orthotomic circle may be any one of the four circles O, R, S, T, which contain the four sets of concyclic foci respectively; and the conic of centres is a conic through the corresponding set of four concyclic foci. We have thus four conics, but the foci of each of them coincide with the nodofoci of the curve, that is, the conics are confocal; that such confocal conics exist has been shown, *ante*, Nos. 78 to 80.

Remark as to the Construction of the Symmetrical Curve—Art. Nos. 180 and 181.

180. It is to be observed that in applying as above the theorem of the variable zomal to the construction of a symmetrical curve, the orthotomic circle made use of was one of the circles R, S, T, not the circle O, which is in this case the axis; in fact, we should then have the conic and the orthotomic circle each of them coinciding with the axis. And the variable circle, *quà* circle having its centre on the axis, cuts the axis at right angles whatever the radius may be; that is, the variable circle is no longer sufficiently determined by the theorem. The curve may nevertheless be constructed as the envelope of a variable circle having its centre on the axis; viz., writing $\mathbf{A}^\circ = (x - az)^2 + y^2 - a''^2z^2$, &c., and starting with the form

$$\sqrt{l\mathbf{A}^\circ} + \sqrt{m\mathbf{B}^\circ} + \sqrt{n\mathbf{C}^\circ} = 0\,,$$

then recurring to the demonstration of the theorem (*ante*, No. 47), the equation of the variable circle is $\mathrm{a}\mathbf{A}^\circ + \mathrm{b}\mathbf{B}^\circ + \mathrm{c}\mathbf{C}^\circ = 0$, where a. b. c are any quantities satisfying $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, or, what is the same thing, taking q an arbitrary parameter, and writing $\frac{l}{\mathrm{a}}=1+q,\ \frac{m}{\mathrm{b}}=1-q,\ \frac{n}{\mathrm{c}}=-2$, the equation of the variable circle is

$$\frac{1}{1+q}\,l\mathbf{A}^\circ+\frac{1}{1-q}\,m\mathbf{B}^\circ-\frac{1}{2}\,n\mathbf{C}^\circ=0.$$

Compare Nos. 118–123 for the like mode of construction of a conic; but it is proper to consider this in a somewhat different form.

181. Assume that the equation of the variable circle is

$$\mathbf{D}^\circ=(x-dz)^2+y^2-d''^2z^2=0;$$

we have therefore identically

$$\mathrm{a}\mathbf{A}^\circ+\mathrm{b}\mathbf{B}^\circ+\mathrm{c}\mathbf{C}^\circ+\mathrm{d}\mathbf{D}^\circ=0,$$

viz., this gives

$$\begin{aligned}\mathrm{a}\ +\mathrm{b}\ +\mathrm{c}\ &=-\mathrm{d},\\ \mathrm{a}a+\mathrm{b}b+\mathrm{c}c&=-\mathrm{d}d,\\ \mathrm{a}(a^2-a''^2)+\mathrm{b}(b^2-b''^2)+\mathrm{c}(c^2-c''^2)&=-\mathrm{d}(d^2-d''^2),\end{aligned}$$

and from these equations we obtain a, b, c equal respectively to given multiples of d; substituting these values in the equation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, d divides out, and we have an equation involving the parameters of the given circles, and also d, d'', the parameters of the variable circle; viz., an equation determining d'', the radius of the variable circle, in terms of d, the co-ordinate of its centre. I consider in particular the case where the given circles are points; that is, where the given equation is

$$\sqrt{l\mathbf{A}}+\sqrt{m\mathbf{B}}+\sqrt{n\mathbf{C}}=0.$$

The equations here are

$$\begin{aligned}\mathrm{a}\ +\mathrm{b}\ +\mathrm{c}\ &=-\mathrm{d}\\ \mathrm{a}a+\mathrm{b}b+\mathrm{c}c&=-\mathrm{d}d\\ \mathrm{a}a^2+\mathrm{b}b^2+\mathrm{c}c^2&=-\mathrm{d}(d^2-d''^2),\end{aligned}$$

and from these we obtain

$$\begin{aligned}\mathrm{a}\,(a-b)\,(a-c)&=-\mathrm{d}\,\big((d-b)\,(d-c)-d''^2\big)\\ \mathrm{b}\,(b-c)\,(b-a)&=-\mathrm{d}\,\big((d-c)\,(d-a)-d''^2\big)\\ \mathrm{c}\,(c-a)\,(c-b)&=-\mathrm{d}\,\big((d-a)\,(d-b)-d''^2\big).\end{aligned}$$

so that the equation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$ becomes

$$\frac{l(a-b)\,(a-c)}{(d-b)\,(d-c)-d''^2}+\frac{m(b-c)\,(b-a)}{(d-c)\,(d-a)-d''^2}+\frac{n(c-a)\,(c-b)}{(d-a)\,(d-b)-d''^2}=0,$$

or, as this is more conveniently written

$$\frac{l}{b-c}\,\frac{1}{(d-b)(d-c)-d''^2}+\frac{m}{c-a}\,\frac{1}{(d-c)(d-a)-d''^2}+\frac{n}{a-b}\,\frac{1}{(d-a)(d-b)-d''^2}=0,$$

viz., considering d, d'' as the abscissa and ordinate of a point on a curve, and representing them by x, y respectively, the equation of this curve is

$$\frac{l}{b-c}\,\frac{1}{(x-b)(x-c)-y^2}+\frac{m}{c-a}\,\frac{1}{(x-c)(x-a)-y^2}+\frac{n}{c-a}\,\frac{1}{(x-a)(x-b)-y^2}=0,$$

which is a certain quartic curve; and we have the original curve

$$\sqrt{l\mathsf{A}}+\sqrt{m\mathsf{B}}+\sqrt{n\mathsf{C}}=0,$$

as the envelope of a variable circle having for its diameter the double ordinate of this quartic curve.

Write for shortness $\frac{l}{b-c}, \frac{m}{c-a}, \frac{n}{a-b} = L, M, N$ respectively, then the equation of the quartic curve may be written

$$\Sigma L\left[(x-a)^2(x-b)(x-c)-y^2(x-a)(2x-b-c)+y^4\right]=0,$$

viz., this is

$$\begin{aligned}\Sigma L\Big[&x(x-a)(x-b)(x-c)\\ &-y^2\big(2x^2-(a+b+c)x+(ab+ac+bc)\big)+y^4\\ &-a(x-a)(x-b)(x-c)+y^2(ax+bc)\Big]=0,\end{aligned}$$

or what is the same thing, the equation is

$$\begin{aligned}(L+M+N)&\left[x(x-a)(x-b)(x-c)-y^2\big(2x^2-(a+b+c)x+ab+ac+bc\big)+y^4\right]\\ &-(La+Mb+Nc)(x-a)(x-b)(x-c)\\ &+y^2\{La+Mb+Nc)x+Lbc+Mca+Nab\}=0.\end{aligned}$$

In the particular case where $L+M+N=0$, that is, where

$$\frac{l}{b-c}+\frac{m}{c-a}+\frac{n}{a-b}=0,$$

the quartic curve becomes a cubic, viz., putting for shortness

$$-\delta=\frac{Lbc+Mca+Nab}{La+Mb+Nc},$$

the equation of the cubic is

$$y^2=\frac{(x-a)(x-b)(x-c)}{x-\delta},$$

viz., this is a cubic curve having three real asymptotes, and a diameter at right angles to one of the asymptotes, and at the inclinations + 45°, — 45° to the other two asymptotes respectively—say that it is a "rectangular" cubic. The relation

$\frac{l}{b-c}+\frac{m}{c-a}+\frac{n}{a-b}=0$ implies that the curve $\sqrt{l\mathsf{A}}+\sqrt{m\mathsf{B}}+\sqrt{n\mathsf{C}}=0$ is a cartesian, and we have thus the theorem that the envelope of a variable circle having for diameter the double ordinate of a rectangular cubic is a cartesian.

I remark that using a particular origin, and writing the equation of the rectangular cubic in the form $y^2 = x^2 - 2mx + \alpha + \frac{2A}{x}$, the equation of the variable circle is

$$(x-d)^2 + y^2 = d^2 - 2md + \alpha + \frac{2A}{d},$$

that is

$$x^2 + y^2 - \alpha - 2d\,(x-m) - \frac{2A}{d} = 0\,,$$

where d is the variable parameter. Forming the derived equation in regard to d, we have

$$x - m = \frac{A}{d^2},$$

and thence

$$x^2 + y^2 - \alpha = \frac{4A}{d},$$

$$(x^2 + y^2 - \alpha)^2 = \frac{16A^2}{d^2} = 16A\,(x-m),$$

that is, the equation of the envelope is $(x^2 + y^2 - \alpha)^2 = 16\,A\,(x-m) = 0$, which is a known form of the equation of a Cartesian.

Focal Formulæ for the General Curve—Art. Nos. 182 and 183.

182. Considering any three circles centres A, B, C, and taking A°, &c., to denote as usual, let the equation of the curve be

$$\sqrt{l\mathsf{A}^{\circ}} + \sqrt{m\mathsf{B}^{\circ}} + \sqrt{n\mathsf{C}^{\circ}} = 0\,;$$

then considering a fourth circle, centre D, a position of the variable circle, and having therefore the same orthotomic circle with the given circles, so that as before

$$\mathrm{a}\mathsf{A}^{\circ} + \mathrm{b}\mathsf{B}^{\circ} + \mathrm{c}\mathsf{C}^{\circ} + \mathrm{d}\mathsf{D}^{\circ} = 0\,,$$

the formulæ No. 47 (changing only U, V, W, T into $\mathsf{A}^{\circ}, \mathsf{B}^{\circ}, \mathsf{C}^{\circ}, \mathsf{D}^{\circ}$) are at once applicable to express the equation of the curve in terms of any three of the four circles A, B, C, D.

In particular, the circles may reduce themselves to the four points A, B, C, D, a set of concyclic foci, and here, the equation being originally given in the form

$$\sqrt{l\mathsf{A}} + \sqrt{m\mathsf{B}} + \sqrt{n\mathsf{C}} = 0,$$

the same formulæ are applicable to express the equation in terms of any three of the four foci.

183. It is to be observed that in this case if the positions of the four foci are given by means of the circular co-ordinates $\left(a, \frac{1}{a}\right)$ &c., which refer to the centre of the circle *ABCD* as origin, and with the radius of this circle taken as unity, then the values of a, b, c, d (*ante*, No. 90), are given in the form adapted to the formulæ of No. 49, viz., we have

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = \alpha(\beta\gamma\delta) : -\beta(\gamma\delta\alpha) : \gamma(\delta\alpha\beta) : -\delta(\alpha\beta\gamma),$$

where $(\beta\gamma\delta) = (\beta-\gamma)(\gamma-\delta)(\delta-\beta)$, &c. The relation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, putting therein $l : m : n = \rho\alpha(\beta-\gamma)^2 : \sigma\beta(\gamma-\alpha)^2 : \tau\gamma(\alpha-\beta)^2$, (or, what is the same thing, taking the equation of the curve to be given in the form $(\beta-\gamma)\sqrt{\rho\alpha\mathbf{A}} + (\gamma-\alpha)\sqrt{\sigma\beta\mathbf{B}} + (\alpha-\beta)\sqrt{\tau\gamma\mathbf{C}} = 0$), becomes

$$\varrho(\beta-\gamma)(\alpha-\delta) + \sigma(\gamma-\alpha)(\beta-\delta) + \tau(\alpha-\beta)(\gamma-\delta) = 0,$$

viz., this equation, considering $\rho, \sigma, \tau, \alpha, \beta, \gamma$ as given, determines the position of the fourth focus *D*, or when *A*, *B*, *C*, *D* are given, it is the relation which must exist between ρ, σ, τ; and the four forms of the equation are

$$\left(\begin{array}{cccc} \cdot\ , & \sqrt{\tau}(\delta-\gamma), & \sqrt{\sigma}(\beta-\delta), & \sqrt{\varrho}(\gamma-\beta) \\ \sqrt{\tau}(\gamma-\delta), & \cdot\ , & \sqrt{\varrho}(\delta-\alpha), & \sqrt{\sigma}(\alpha-\gamma) \\ \sqrt{\sigma}(\delta-\beta), & \sqrt{\varrho}(\alpha-\delta), & \cdot\ , & \sqrt{\tau}(\beta-\alpha) \\ \sqrt{\varrho}(\beta-\gamma), & \sqrt{\sigma}(\gamma-\alpha), & \sqrt{\tau}(\alpha-\beta), & \cdot \end{array}\right)(\sqrt{\alpha\mathbf{A}}, \sqrt{\beta\mathbf{B}}, \sqrt{\gamma\mathbf{C}}, \sqrt{\delta\mathbf{D}}) = 0,$$

viz., the curve is represented by means of any one of these four equations involving each of them three out of the four given foci *A*, *B*, *C*, *D*.

Case of the Circular Cubic—Art. Nos. 184 and 185.

184. In the case of a circular cubic, we must have

$$\begin{array}{llll} \varrho(\beta-\gamma)(\alpha-\delta) & +\ \sigma(\gamma-\alpha)(\beta-\delta) & +\ \tau(\alpha-\beta)(\gamma-\delta) & = 0, \\ \sqrt{\alpha\varrho}(\beta-\gamma) & +\ \sqrt{\beta\sigma}(\gamma-\alpha) & +\ \sqrt{\gamma\tau}(\alpha-\beta) & = 0, \end{array}$$

which, when the foci *A*, *B*, *C*, *D* are given, determine the values of $\rho : \sigma : \tau$ in order that the curve may be a circular cubic. We see at once that there are two sets of values, and consequently two circular cubics having each of them the given points *A*, *B*, *C*, *D* for a set of concyclic foci. The two systems may be written

$$\sqrt{\varrho} : \sqrt{\sigma} : \sqrt{\tau} = \sqrt{\alpha\delta} - \sqrt{\beta\gamma} : \sqrt{\beta\delta} - \sqrt{\gamma\alpha} : \sqrt{\gamma\delta} - \sqrt{\alpha\beta},$$

viz., it being understood that $\sqrt{\alpha\delta}$ means $\sqrt{\alpha}\ \sqrt{\delta}$, &c., then, according as $\sqrt{\delta}$ has one or other of its two opposite values, we have one or other of the two systems of values of $\rho:\sigma:\tau$. To verify this, observe that writing the equation under the form

$$\sqrt{\alpha\varrho}:\sqrt{\beta\sigma}:\sqrt{\gamma\tau}=\alpha\sqrt{\delta}-\sqrt{\alpha\beta\gamma}:\beta\sqrt{\delta}-\sqrt{\alpha\beta\gamma}:\gamma\sqrt{\delta}-\sqrt{\alpha\beta\gamma},$$

the second equation is verified; and that writing them under the form

$$\varrho:\sigma:\tau=-(\beta+\gamma)(\alpha+\delta)+M:-(\gamma+\alpha)(\beta+\delta)+M:-(\alpha+\beta)(\gamma+\delta)+M,$$

where

$$M=\beta\gamma+\alpha\delta+\gamma\alpha+\beta\delta+\alpha\beta+\gamma\delta-2\sqrt{\alpha\beta\gamma\delta},$$

the second equation is also verified.

185. If we assume for a moment $\alpha=\cos a+i\sin a=e^{ia}$, &c., viz., if a, b, c, d be the inclinations to any fixed line of the radii through A, B, C, D respectively, then we have

$$\begin{aligned}\sqrt{\alpha\delta}\pm\sqrt{\beta\gamma}&=e^{\frac{1}{4}(a+b+c+d)i}\left\{e^{\frac{1}{4}(a+d-b-c)i}\pm e^{-\frac{1}{4}(a+d-b-c)i}\right\},\\ \sqrt{\alpha}(\beta-\gamma)&=e^{\frac{1}{2}(a+b+c)i}\left\{e^{\frac{1}{2}(b-c)i}-e^{-\frac{1}{2}(b-c)i}\right\},\end{aligned}$$

and thence

$$\begin{aligned}\sqrt{\alpha\varrho}(\beta-\gamma):\sqrt{\beta\sigma}(\gamma-\alpha):\sqrt{\gamma\tau}(\alpha-\beta)&=\cos\tfrac{1}{4}(a+d-b-c)\sin\tfrac{1}{2}(b-c)\\ &:\cos\tfrac{1}{4}(b+d-c-a)\sin\tfrac{1}{2}(c-a)\\ &:\cos\tfrac{1}{4}(a+d-a-b)\sin\tfrac{1}{2}(a-b);\end{aligned}$$

or else

$$\begin{aligned}&=\sin\tfrac{1}{4}(a+d-b-c)\sin\tfrac{1}{2}(b-c)\\ &:\sin\tfrac{1}{4}(b+d-c-a)\sin\tfrac{1}{2}(c-a)\\ &:\sin\tfrac{1}{4}(c+d-a-b)\sin\tfrac{1}{2}(a-b).\end{aligned}$$

Putting in these formulæ,

$$\begin{array}{lcl}\tfrac{1}{4}(a-b-c)=A, & \text{then we have} & B-C=\tfrac{1}{2}(b-c),\\ \tfrac{1}{4}(b-c-a)=B, & ,, & C-A=\tfrac{1}{2}(c-a),\\ \tfrac{1}{4}(c-a-b)=C, & ,, & A-B=\tfrac{1}{2}(a-b);\end{array}$$

and for either set of values the verification of the relation

$$\sqrt{\alpha\varrho}(\beta-\gamma)+\sqrt{\beta\sigma}(\gamma-\alpha)+\sqrt{\gamma\tau}(\alpha-\beta)=0,$$

will depend on the two identical equations

$$\begin{aligned}\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C\sin(A-B)&=0,\\ \cos A\sin(B-C)+\cos B\sin(C-A)+\cos C\sin(A-B)&=0:\end{aligned}$$

although the foregoing solution for the case of a circular cubic is the most elegant one, I will presently return to the question and give the solution in a different form.

Focal Formulæ for the Symmetrical Curve—Art. No. 186.

186. In the symmetrical case, where the foci A, B, C, D are on a line, then if, as usual, a, b, c, d denote the distances from a fixed point, we have the ex-

pressions of (a, b, c, d) in a form adapted to the formulæ of No. 49, viz.,

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = (b-c)(c-d)(d-b) : -(c-d)(d-a)(a-c) : (d-a)(a-b)(b-d) : -(a-b)(b-c)(c-a)\,,$$

so that, assuming

$$l : m : n = \varrho(b-c)^2 : \sigma(c-a)^2 : \tau(a-b)^2\,,$$

the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0\,,$ becomes

$$\varrho(b-c)(a-d) + \sigma(c-a)(b-d) + \tau(a-b)(c-d) = 0\,,$$

and the equation of the curve may be presented under any one of the four forms

$$\left(\begin{array}{cccc} \cdot\,, & \sqrt{\tau}(d-c)\,, & \sqrt{\sigma}(b-d)\,, & \sqrt{\varrho}(c-b) \\ \sqrt{\tau}(c-d)\,, & \cdot\,, & \sqrt{\varrho}(d-a)\,, & \sqrt{\sigma}(a-c) \\ \sqrt{\sigma}(d-b)\,, & \sqrt{\varrho}(a-d)\,, & \cdot\,, & \sqrt{\tau}(b-a) \\ \sqrt{\varrho}(b-c)\,, & \sqrt{\sigma}(c-a)\,, & \sqrt{\tau}(a-b)\,, & \cdot \end{array}\right)\left(\sqrt{\mathrm{A}}, \sqrt{\mathrm{B}}, \sqrt{\mathrm{C}}, \sqrt{\mathrm{D}}\right) = 0\,.$$

Case of the Symmetrical Circular Cubic—Art. No. 187.

187. For a circular cubic we must have

$$\begin{aligned} \varrho(b-c)(a-d) + \sigma(c-a)(b-d) + \tau(a-b)(c-d) &= 0 \\ \sqrt{\varrho}(b-c) + \sqrt{\sigma}(c-a) + \sqrt{\tau}(a-b) &= 0\,. \end{aligned}$$

These equations give $\sqrt{\rho} : \sqrt{\sigma} : \sqrt{\tau} = 1 : 1 : 1$ (values which obviously satisfy the two equations), or else

$$\sqrt{\varrho} : \sqrt{\sigma} : \sqrt{\varrho} = a + d - b - c : b + d - c - a : c + d - a - b\,.$$

In fact, these values obviously satisfy the second equation; and to see that they satisfy the first equation, we have only to write them under the form

$$\varrho : \sigma : \tau = M - 4(b+c)(a+d) : M - 4(c+a)(b+d) : M - 4(a+b)(c+d)\,,$$

where $M = (a+b+c+d)^2$. The first set gives for the curve

$$(b-c)\sqrt{\mathrm{A}} + (c-a)\sqrt{\mathrm{B}} + (a-b)\sqrt{\mathrm{C}} = 0\,,$$

but this contains the line $z = 0$ not once only, but twice; it in fact is $(y^2 = 0)$, the axis taken twice; the only proper cubic with the foci A, B, C, D *in lineâ* is therefore

$$(b-c)(a+d-b-c)\sqrt{\mathrm{A}} + (c-a)(b+d-c-a)\sqrt{\mathrm{B}} + (a-b)(c+d-a-b)\sqrt{\mathrm{C}} = 0\,,$$

the equation of which is, of course, expressible in each of the other three forms.

Case of the General Circular Cubit—Art. Nos. 188 to 192.

188. Returning to the general case of the circular cubic, the lines BC, AD

meet in R, and if we denote by a_1, b_1, c_1, d_1, the distances from R of the four points respectively, so that $b_1 c_1 = a_1 d_1 = \text{rad.}^2 R$, then observing that a, b, c, d are proportional to the triangles BCD, CDA, DAB, ABC, with signs such that $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d} = 0$, we find

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = -d_1(b_1 - c_1) : c_1(a_1 - d_1) : -b_1(a_1 - d_1) : a_1(b_1 - c_1);$$

and this being so, the equations $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, $\sqrt{l} + \sqrt{m} + \sqrt{n} = 0$, give two systems of values of $\sqrt{l} : \sqrt{m} : \sqrt{n}$, viz., these are

$$\sqrt{l} : \sqrt{m} : \sqrt{n} = b_1 - c_1 : c_1 - a_1 : a_1 - b_1,$$

and

$$= b_1 - c_1 : c_1 + a_1 : -a_1 - b_1.$$

(To verify this, observe that for the first set we have

$$\begin{aligned} \frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} &= \frac{(b_1 - c_1)^2}{-d_1(b_1 - c_1)} + \frac{(c_1 - a_1)^2}{c_1(a_1 - d_1)} + \frac{(a_1 - b_1)^2}{-b_1(a_1 - d_1)} \\ &= \frac{b_1 - c_1}{-d_1} + \frac{1}{a_1 - d_1}\left(c_1 + \frac{a_1^2}{c_1} - b_1 - \frac{a_1^2}{b_1}\right) \\ &= \frac{b_1 - c_1}{-d_1} + \frac{b_1 - c_1}{a_1 - d_1}\left(\frac{a_1^2}{b_1 c_1} - 1\right) \\ &= -\frac{b_1 - c_1}{d_1} + \frac{b_1 - c_1}{a_1 - d_1}\left(\frac{a_1}{d_1} - 1\right) = 0; \end{aligned}$$

and the like as regards the second set).

189. These values of $\sqrt{l} : \sqrt{m} : \sqrt{n}$ give the equations of the two circular cubics with the foci (A, B, C, D), the equation of each of them under a fourfold form, viz., we have

$$\left(\begin{array}{cccc} \cdot\ , & d_1 - c_1\ , & b_1 - d_1\ , & c_1 - b_1 \\ c_1 - d_1\ , & \cdot\ , & d_1 - a_1\ , & a_1 - c_1 \\ d_1 - b_1\ , & a_1 - d_1\ , & b_1 - a_1\ , & b_1 - a_1 \\ b_1 - c_1\ , & c_1 - a_1\ , & a_1 - b_1\ , & \cdot \end{array}\right)(\sqrt{\mathsf{A}}, \sqrt{\mathsf{B}}, \sqrt{\mathsf{C}}, \sqrt{\mathsf{D}}) = 0 \quad \text{(first curve)},$$

and

$$\left(\begin{array}{cccc} \cdot\ , & -c_1 - d_1\ , & d_1 + b_1\ , & -b_1 + c_1 \\ d_1 + c_1\ , & \cdot\ , & a_1 - d_1\ , & c_1 - a_1 \\ -b_1 - d_1\ , & d_1 - a_1\ , & \cdot\ , & a_1 + b_1 \\ b_1 - c_1\ , & c_1 + a_1\ , & -a_1 - b_1\ , & \cdot \end{array}\right)(\sqrt{\mathsf{A}}, \sqrt{\mathsf{B}}, \sqrt{\mathsf{C}}, \sqrt{\mathsf{D}}) = 0 \quad \text{(second curve)}.$$

190. Similarly CA and BD meet in S, and if we denote by a_2, b_2, c_2, d_2 the distances from S of the four points respectively, so that $c_2 a_2 = b_2 d_2 = \text{rad.}^2 S$ (observe that if as usual A, B, C, D are taken in order on the circle O, then A, C are on opposite sides of S, and similarly B, D are on opposite sides of S, so that

taking a_2, b_2 positive c_2, d_2 will be negative) we have

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d} = c_2(b_2-d_2):d_2(c_2-a_2):-a_2(b_2-d_2):-b_2(c_2-a_2)\,,$$

and then the equations $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, are satisfied by the two sets of values

$$\sqrt{l}\ \sqrt{m}:\sqrt{n} = \quad b_2-c_2:c_2-a_2:a_2-b_2\,,$$

and

$$= -b_2-c_2:c_2-a_2:a_2+b_2\,,$$

and we have the equations of the same two cubic curves, each equation under a fourfold form, viz., these are

$$\left(\begin{array}{cccc} \cdot\ , & -c_2+d_2\,, & -d_2+b_2\,, & -b_2+c_2 \\ c_2-d_2\,, & \cdot\ , & d_2-a_2\,, & -c_2+a_2 \\ -b_2+d_2\,, & a_2-d_2\,, & \cdot\ , & -a_2+b_2 \\ b_2-c_2\,, & c_2-a_2\,, & a_2-b_2\,, & \cdot \end{array}\right)(\sqrt{\mathsf{A}},\sqrt{\mathsf{B}},\sqrt{\mathsf{C}},\sqrt{\mathsf{D}})=0$$

(first curve),

and

$$\left(\begin{array}{cccc} \cdot\ , & c_2+d_2\,, & -d_2+b_2\,, & -b_2-c_2 \\ -d_2+c_2\,, & \cdot\ , & a_2+d_2, & c_2-a_2 \\ -b_2+d_2, & -d_2-a_2\,, & \cdot\ , & a_2+b_2 \\ b_2+c_2\,, & -c_2+a_2\,, & -a_2-b_2\,, & \cdot \end{array}\right)(\sqrt{\mathsf{A}},\sqrt{\mathsf{B}},\sqrt{\mathsf{C}},\sqrt{\mathsf{D}})=0\,.$$

(second curve).

191. And again AB and CD meet in T, and denoting by a_3, b_3, c_3, d_3 the distances from T of the four points respectively, so that $a_3b_3=c_3d_3=\text{rad.}\,{}^2T$, we have

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d} = b_3(c_3-d_3):-a_3(c_3-d_3):-d_3(a_3-b_3):c_3(a_3-b_3)\,.$$

The equations $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, then give for $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$ two sets of values, viz., these are

$$\sqrt{l}:\sqrt{m}:\sqrt{n} = b_3-c_3: \quad c_3-a_3:a_3-b_3\,,$$

and

$$= b_3+c_3:-c_3-a_3:a_3-b_3\,;$$

and we again obtain the equations of the two cubics, each equation under a fourfold form, viz., these are

$$\left(\begin{array}{cccc} \cdot\ , & -c_3+d_3\,, & -d_3+b_3\,, & c_3-b_3 \\ -d_3+c_3\,, & \cdot\ , & -a_3+d_3\,, & a_3-c_3 \\ -b_3+d_3\,, & -d_3+a_3\,, & \cdot\ , & b_3-a_3 \\ b_3-c_3\,, & c_3-a_3\,, & a_3-b_3\,, & \cdot \end{array}\right)(\sqrt{\mathsf{A}},\sqrt{\mathsf{B}},\sqrt{\mathsf{C}},\sqrt{\mathsf{D}})=0\,,$$

and

$$\left(\begin{array}{cccc} \cdot\ , & c_3-d_3\,, & d_3+b_3\,, & -c_3-b_3 \\ d_3-c_3\,, & \cdot\ , & -a_3-d_3\,, & a_3+c_3 \\ b_3-d_3\,, & d_3+a_3\,, & \cdot\ , & b_3-a_3 \\ b_3+c_3\,, & -c_3-a_3\,, & a_3-b_3\,, & \cdot \end{array}\right)(\sqrt{\mathsf{A}},\sqrt{\mathsf{B}},\sqrt{\mathsf{C}},\sqrt{\mathsf{D}})=0\,.$$

192. The three systems have been obtained independently, but they may of course be derived each from any other of them: to show how this is, recollecting that we have

$$\begin{aligned} RA, RB, RC, RD &= a_1, b_1, c_1, d_1, \\ SA, SB, SC, SD &= a_2, b_2, -c_2, -d_2, \\ TA, TB, TC, TD &= a_3, b_3, c_3, d_3; \end{aligned}$$

then to compare

$$(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2);$$

similar triangles

$$\begin{aligned} &SBC \text{ give } b_1 - c_1 : -c_2 : b_2, \\ &SAD \quad = a_1 - d_1 : -d_2 : a_2, \end{aligned}$$

and similar triangles

$$\begin{aligned} &RAC \text{ give } a_2 - c_2 : c_1 : a_1, \\ &RBD \quad = b_2 - d_2 : d_1 : b_1; \end{aligned}$$

using these equations to determine the ratios of a_2, b_2, c_2, d_2 we have

$$\frac{a_2 - c_2}{b_2 - d_2} = \frac{c_1}{d_1}, \text{ or } d_1 a_2 - d_1 c_2 - c_1 b_2 + c_1 d_2 = 0;$$

that is

$$b_2 \left\{ -c_1 + d_1 \frac{a_1 - d_1}{b_1 - c_1} \right\} + c_2 \left\{ -d_1 + c_1 \frac{a_1 - d_1}{b_1 - c_1} \right\} = 0;$$

and hence

$$b_2(-b_1 c_1 + c_1^2 + a_1 d_1 - d_1^2) + c_2(-b_1 d_1 + c_1 d_1 + a_1 c_1 - c_1 d_1) = 0,$$

that is

$$b_2(c_1^2 - d_1^2) + c_2(a_1 c_1 - b_1 d_1) = 0,$$

but

$$a_1 c_1 - b_1 d_1 = \frac{b_1}{d_1}(c_1^2 - d_1^2),$$

or the equation gives $b_2 + \frac{b_1}{d_1} c_2 = 0$, or say $b_2 : c_2 = b_1 : -d_1$, and this with $\frac{b_1 - c_1}{a_1 - d_1} = \frac{c_2}{d_2} = \frac{b_2}{a_2}$, gives all the ratios, or we have

$$a_2 : b_2 : c_2 : d_2 = b_1(a_1 - d_1) : b_1(b_1 - c_1) : -d_1(a_1 - d_1) : -d_1(b_1 - c_1).$$

We have then for example

$$b_2 - c_2 : c_2 - a_2 : a_2 - b_2 = b_1 - c_1 : c_1 - a_1 : a_1 - {}_1b; \text{ \&c.},$$

showing the identity of the forms in (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2).

Transformation to a New Set of Concyclic Foci.—Art. No. 193.

193. Consider the equation

$$\sqrt{l\mathsf{A}} + \sqrt{m\mathsf{B}} + \sqrt{n\mathsf{C}} = 0,$$

which refers to the foci A, B, C, and taking D the fourth concyclic focus, let

(A_1, D_1) be the antipoints of (A, D) and (B_1, C_1) the antipoints of (B, C); so that (A_1, B_1, C_1, D_1) are another set of concyclic foci. We have $\mathsf{B}_1 . \mathsf{C}_1 = \mathsf{B} . \mathsf{C}$, and it appears, *ante* No. 104, that we can find l_1, m_1, n_1, such that identically

$$-l\mathsf{A} + m\mathsf{B} + n\mathsf{C} = -l_1\mathsf{A}_1 + m_1\mathsf{B}_1 + n_1\mathsf{C}_1$$

and that $m_1 n_1 = m n$. The equation of the curve gives

$$-l\mathsf{A} + m\mathsf{B} + n\mathsf{C} + 2\sqrt{mn\mathsf{BC}} = 0\,,$$

we have therefore

$$-l_1\mathsf{A}_1 + m_1\mathsf{B}_1 + n_1\mathsf{C} + 2\sqrt{m_1n_1\mathsf{B}_1\mathsf{C}_1} = 0\,,$$

that is,

$$\sqrt{l_1\mathsf{A}_1} + \sqrt{m_1\mathsf{B}_1} + \sqrt{n_1\mathsf{C}_1} = 0\,,$$

viz., this is the equation of the curve expressed in terms of the concyclic foci A_1, B_1, C_1.

The Tetrazomal Curve, Decomposable or Indecomposable.—Art. No. 194.

194. I consider the tetrazomal curve

$$\sqrt{l\mathsf{A}^\circ} + \sqrt{m\mathsf{B}^\circ} + \sqrt{n\mathsf{C}^\circ} + \sqrt{p\mathsf{D}^\circ} = 0\,,$$

where the zomals are circles described about any given points A, B, C, D as centres.

There is not, in general, any identical equation $\mathrm{a}\mathsf{A}^\circ + \mathrm{b}\mathsf{B}^\circ + \mathrm{c}\mathsf{C}^\circ + \mathrm{d}\mathsf{D}^\circ = 0$, but when such relation exists, and when we have also $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0$, then the curve breaks up into two trizomals. When the conditions in question do not subsist, the curve is indecomposable. But there may exist between l, m, n, p relations in virtue of which a branch or branches ideally contain ($z^a = 0$) the line infinity a certain number of times, and which thus cause a depression in the order of the curve. The several cases are as follows:—

Cases of the Indecomposable Curve.—Art. No. 195.

195. I. The general case; l, m, n, p not subjected to any condition. The curve is here of the order $= 8$; it has a quadruple point at each of the points I, J (and there is consequently no other point at infinity); it is touched four times by each of the circles A, B, C, D; and it has six nodes, viz., these are the intersections of the pairs of circles

$$\begin{aligned}
&\sqrt{m\mathsf{B}^\circ} + \sqrt{n\mathsf{C}^\circ} = 0, \quad \sqrt{l\mathsf{A}^\circ} + \sqrt{p\mathsf{D}^\circ} = 0\,,\\
&\sqrt{n\mathsf{C}^\circ} + \sqrt{l\mathsf{A}^\circ} = 0, \quad \sqrt{m\mathsf{B}^\circ} + \sqrt{p\mathsf{D}^\circ} = 0\,,\\
&\sqrt{l\mathsf{A}^\circ} + \sqrt{m\mathsf{B}^\circ} = 0, \quad \sqrt{n\mathsf{C}^\circ} + \sqrt{p\mathsf{D}^\circ} = 0\,;
\end{aligned}$$

the number of dps. is $6 + 2 . 6, = 18$, and there are no cusps, hence the class is $= 20$, and the deficiency is $= 3$.

II. We may have

$$\sqrt{l} + \sqrt{m} + \sqrt{n} + \sqrt{p} = 0\,;$$

there is in this case a single branch ideally containing ($z = 0$) the line infinity; the order is $= 7$. Each of the points I, J is a triple point, there is consequently one other point at infinity; viz., this is a real point, or the curve has a real asymptote. There are 6 nodes as before; dps. are $6 + 2\,.\,3, = 12$; class $= 18$, deficiency $= 3$.

III. We may have

$$\sqrt{l} + \sqrt{m} = 0,\ \sqrt{n} + \sqrt{p} = 0\,;$$

there are then two branches each ideally containing ($z = 0$) the line infinity; the order is $= 6$. Each of the points I, J is a double point, and there are therefore two more points at infinity. These may be real or imaginary; viz., the curve may have (besides the asymptotes at I, J) two real or imaginary asymptotes. The circles $\sqrt{l\mathsf{A}} + \sqrt{m\mathsf{B}} = 0$, $\sqrt{n\mathsf{C}} + \sqrt{p\mathsf{D}} = 0$, each contain ($z = 0$) the line infinity, or they reduce themselves to two lines, so that in place of two nodes we have a single node at the intersection of these lines; number of nodes is $= 5$. Hence dps. are $5 + 2\,.\,1, = 7$. Class is $= 16$, deficiency $= 3$.

IV. We may have

$$\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = \mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d}$$

there is here a single branch containing ($z^2 = 0$) the line infinity twice; the order is $= 6$. Each of the points I, J is a double point, and there are therefore two more points at infinity, that is (besides the asymptotes at I, J), there are two (real or imaginary) asymptotes. The number of nodes, as in the general case, is $= 6$. Hence dps. are $6 + 2\,.\,1, = 8$; class is $= 14$; deficiency $= 2$.

I notice the included particular case where the circles reduce themselves to their centres; viz., we have here the curve

$$\mathrm{a}\sqrt{\mathsf{A}} + \mathrm{b}\sqrt{\mathsf{B}} + \mathrm{c}\sqrt{\mathsf{C}} + \mathrm{d}\sqrt{\mathsf{D}} = 0\,,$$

which (see *ante* No. 93) is in fact the curve which is the locus of the foci of the conics which pass through the four points A, B, C, D. It is at present assumed that the four points are not a circle; this case will be considered *post* No. 199. If we have BC, AD meeting in R; CA, BD in S, and AB, CD in T, then these points R, S, T are three of the six nodes. In fact, writing down the equations of the two circles

$$\mathrm{b}\sqrt{\mathsf{B}} + \mathrm{c}\sqrt{\mathsf{C}} = 0\,,\ \mathrm{a}\sqrt{\mathsf{A}} + \mathrm{d}\sqrt{\mathsf{D}} = 0\,,$$

and observing that when the current point is taken at R, we have $\mathsf{B} : \mathsf{C} = \overline{RB}^2 : \overline{RC}^2 = (BAD)^2 : (CAD)^2 = \mathrm{c}^2 : \mathrm{b}^2$, and similarly $\mathsf{A} : \mathsf{D} = \overline{RA}^2 : \overline{RD}^2 = (ABC)^2 : (DBC)^2 = \mathrm{d}^2 : \mathrm{a}^2$, we see that each of the two circles passes through the point *R*, or this point is a node. Similarly, the points *S* and *T* are each of them a node.

V. If
$$\sqrt{l} = \sqrt{m} = \sqrt{n} = \sqrt{p},$$
there are here three branches, each ideally containing ($z = 0$) the line infinity; the order is thus $= 5$. Each of the points I, J is an ordinary point on the curve; there are besides at infinity three points, all real, or one real and two imaginary; that is (besides the asymptotes at I, J) there are three asymptotes, all real, or one real and two imaginary. Each of the circles $\sqrt{\mathsf{A}} + \sqrt{\mathsf{B}} = 0$, &c., contains the line infinity, and is thus reduced to a line; the number of nodes is therefore $= 3$. Hence also, dps. $= 3$; class $= 14$; deficiency $= 3$.

Cases of the Indecomposable Curve, the Centres being in a Line.—Art. No. 196.

196. There are some peculiarities in the case where the centres A, B, C, D are on a line; taking as usual (a, b, c, d) for the x-co-ordinates or distances of the four centres from a fixed point on the line, I enumerate the cases as follows :—

I. No relation between l, m, n, p; corresponds to I. *supra*.

II. $\sqrt{l} + \sqrt{m} + \sqrt{n} + \sqrt{p} = 0$; corresponds to II. *supra*.

III. $\sqrt{l} + \sqrt{m} = 0, \sqrt{n} + \sqrt{p} = 0$; corresponds to III. *supra*.

IV. $\sqrt{l} + \sqrt{m} + \sqrt{n} + \sqrt{p} = 0,\ a\sqrt{l} + b\sqrt{m} + c\sqrt{n} + d\sqrt{p} = 0$; corresponds to IV. *supra*, viz., there is a branch ideally containing ($z^2 = 0$) the line infinity twice. But, observe that whereas in IV. *supra*, in order that this might be so, it was necessary to impose on l, m, n, p three conditions giving the definite systems of values $\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = \mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d}$, in the present case only two conditions are imposed, so that a single arbitrary parameter is left.

V. $\sqrt{l} = \sqrt{m} = \sqrt{n} = \sqrt{p}$; corresponds to V. *supra*.

VI. $\sqrt{l} + \sqrt{m} = 0,\ \sqrt{n} + \sqrt{p} = 0,\ a\sqrt{l} + b\sqrt{m} + c\sqrt{n} + d\sqrt{p} = 0$, or what is the same thing, $\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = c - d : d - c : b - a : a - b$; the equation is thus $(c-d)(\sqrt{\mathsf{A}^\circ} - \sqrt{\mathsf{B}^\circ}) - (a-b)(\sqrt{\mathsf{A}^\circ} - \sqrt{\mathsf{B}^\circ}) = 0$. There is here one branch ideally containing ($z^2 = 0$) the line infinity twice, and another branch ideally containing ($z = 0$) the line infinity once; order is $= 5$. Each of the points I, J is an ordinary point on the curve, the remaining points at infinity are a node ($\mathsf{A}^\circ = \mathsf{B}^\circ, \mathsf{C}^\circ = \mathsf{D}^\circ$), as presently mentioned, counting as three points, viz., one branch has for its tangent the line infinity, and the other branch has for its tangent a line perpendicular to the axis; or what is the same thing, there is a hyperbolic branch having an asymptote perpendicular to the axis, and a parabolic branch ultimately perpendicular to the axis. The number of nodes is $= 5$, viz., there is the node $\mathsf{A}^\circ = \mathsf{B}^\circ, \mathsf{C}^\circ = \mathsf{D}^\circ$ just referred to; and the two pairs of nodes $((c-d)\sqrt{\mathsf{A}^\circ} - (a-b)\sqrt{\mathsf{C}^\circ} = 0,\ -(c-d)\sqrt{\mathsf{B}^\circ} + (a-b)\sqrt{\mathsf{D}^\circ} = 0)$ and $(c-d)\sqrt{\mathsf{A}^\circ} + (a-b)\sqrt{\mathsf{D}^\circ} = 0,\ (c-d)\sqrt{\mathsf{B}^\circ} + (a-b)\sqrt{\mathsf{C}^\circ} = 0)$, each pair symmetrically situate in regard to the axis. Hence also dps. = 5; class = 10; deficiency = 1.

And there is apparently a seventh case, which, however, I exclude from the present investigation, viz., this would be if we had

$$\left(\begin{array}{cccc} 1\,, & 1\,, & 1\,, & 1\,, \\ a\,, & b\,, & c\,, & d\,, \\ a^2, & b^2, & c^2, & d^2, \\ a'^2, & b''^2, & c'^2, & d''^2, \end{array}\right)\!\!\left(\sqrt{l},\ \sqrt{m},\ \sqrt{n},\ \sqrt{p}\right) = 0\,,$$

that is, a, b, c, d denoting as before, if we had

$$\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = \mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d}, \quad \text{and} \quad \mathrm{a}a''^2 + \mathrm{b}b''^2 + \mathrm{c}c''^2 + \mathrm{d}d''^2 = 0\,.$$

For observe that in this case we have

$$\mathrm{a}\mathbf{A}^\circ + \mathrm{b}\mathbf{B}^\circ + \mathrm{c}\mathbf{C}^\circ + \mathrm{d}\mathbf{D}^\circ = 0\,, \quad \text{and} \quad \frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0\,;$$

that is, the supposition in question belongs to the decomposable case.

The Decomposable Curve—Art. No. 197.

197. We have next to consider the decomposable case, viz., when we have

$$\mathrm{a}\mathbf{A}^\circ + \mathrm{b}\mathbf{B}^\circ + \mathrm{c}\mathbf{C}^\circ + \mathrm{d}\mathbf{D}^\circ = 0\,;$$

see *ante*, Nos. 87 *et seq.*—it there appears that (unless the centres A, B, C, D are in a line) the condition signifies that the four circles have a common orthotomic circle; and when we have also

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0\,.$$

The formulæ for the decomposition are given *ante*, Nos. 42 *et seq.* Writing therein $\mathbf{A}^\circ$, $\mathbf{B}^\circ$, $\mathbf{C}^\circ$, $\mathbf{D}^\circ$ in place of U, V, W, T respectively, it thereby appears that the tetrazomal curve $\sqrt{l\mathbf{A}^\circ} + \sqrt{m\mathbf{B}^\circ} + \sqrt{n\mathbf{C}^\circ} + \sqrt{p\mathbf{D}^\circ} = 0$, breaks up into the two trizomal curves

$$\sqrt{l_1\mathbf{A}^\circ} + \sqrt{m_1\mathbf{B}^\circ} + \sqrt{n_1\mathbf{C}^\circ} = 0\,, \quad \sqrt{l_2\mathbf{A}^\circ} + \sqrt{m_2\mathbf{B}^\circ} + \sqrt{n_2\mathbf{C}^\circ} = 0\,,$$

where

$$\sqrt{l_1} = \sqrt{l} + \frac{\mathrm{a}}{\mathrm{d}}\,\frac{p}{\sqrt{l}}\,, \qquad \sqrt{l_2} = \sqrt{l} + \frac{\mathrm{a}}{\mathrm{d}}\,\frac{p}{\sqrt{l}}\,,$$

$$\sqrt{m_1} = \sqrt{m} - \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\,\frac{p}{l}}\,\mathrm{b}\sqrt{n}\,, \qquad \sqrt{m_2} = \sqrt{m} + \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\,\frac{p}{l}}\,\mathrm{b}\sqrt{n}\,,$$

$$\sqrt{n_1} = \sqrt{n_1} + \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\,\frac{p}{l}}\,\mathrm{c}\sqrt{m}\,, \qquad \sqrt{n_2} = \sqrt{n} - \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\,\frac{p}{l}}\,\mathrm{c}\sqrt{m}\,,$$

and where we have

$$\frac{l_1}{\mathrm{a}} + \frac{m_1}{\mathrm{b}} + \frac{n_1}{\mathrm{c}} = 0\,, \qquad \frac{l_2}{\mathrm{a}} + \frac{m_2}{\mathrm{b}} + \frac{n_2}{\mathrm{c}} = 0\,.$$

Cases of the Decomposable Curve, Centres not in a line—Art. Nos. 198 to 203.

198. I assume, in the first instance, that the centres of the circles are not in a line; we have the following cases:—

I. No further relation between l, m, n, p; the order of the tetrazomal is $= 8$ the order of each of the trizomals is $= 4$, that is each of them is a bicircular quartic.

II. $\sqrt{l} + \sqrt{m} + \sqrt{n} + \sqrt{p} = 0$; the order of the tetrazomal is $= 7$, that of one of the trizomals must be $= 3$.

To verify this, observe that we have

$$\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = \sqrt{l} + \sqrt{m} + \sqrt{n} + \frac{\mathrm{a}p}{\mathrm{d}\sqrt{l}} + \frac{\sqrt{p}}{\sqrt{l}}\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}(\mathrm{c}\sqrt{m} - \mathrm{b}\sqrt{n}),$$

or substituting for $\sqrt{l} + \sqrt{m} + \sqrt{n}$ the value $-\sqrt{p}$, this is

$$= \frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left\{\mathrm{a}\sqrt{p} - \mathrm{d}\sqrt{l} + \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}(\mathrm{c}\sqrt{m} - \mathrm{b}\sqrt{n})\right\},$$

and similarly for $\sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2}$, the only change being in the sign of the radical $\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}$. But from the two conditions satisfied by l, m, n, p it is easy to deduce

$$(\mathrm{a}\sqrt{p} - \mathrm{d}\sqrt{l})^2 - \frac{\mathrm{ad}}{\mathrm{bc}}(\mathrm{c}\sqrt{m} - \mathrm{b}\sqrt{n})^2 = 0,$$

and hence one or other of the two functions

$$\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1}, \quad \sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2} \text{ is } = 0;$$

that is, one of the trizomal curves is a cubic.

III. $\sqrt{l} + \sqrt{p} = 0$, $\sqrt{m} + \sqrt{n} = 0$; order of the tetrazomal is $= 6$; and hence order of each of the trizomals is $= 3$. To verify this, observe that here

$$l\left(\frac{1}{\mathrm{a}} + \frac{1}{\mathrm{d}}\right) + m\left(\frac{1}{b} + \frac{1}{\mathrm{c}}\right) = 0,$$

which since $\mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} = 0$, gives $\frac{l}{m} = \frac{\mathrm{ad}}{\mathrm{bc}}$; so that properly fixing the sign of the radical, we may write $\sqrt{l} + \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m} = 0$. We have then

$$\sqrt{l_1} = \frac{\mathrm{a} + \mathrm{d}}{\mathrm{d}}\sqrt{l}, \qquad \sqrt{m_1} + \sqrt{n_1} = \sqrt{\frac{\mathrm{a}}{\mathrm{abc}}}(\mathrm{b} + \mathrm{c})\sqrt{m};$$

which last equation, using $\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}$ to denote as above, but properly selecting the signification of ±, may be written

$$\sqrt{m_1}+\sqrt{n_1}=\pm\frac{\mathrm{b}+\mathrm{c}}{\mathrm{d}}\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}\,.$$

Hence

$$\sqrt{l_1}\mp(\sqrt{m_1}+\sqrt{n_1})=\frac{1}{\mathrm{a}}\left\{(\mathrm{a}+\mathrm{d})\sqrt{l}+(\mathrm{b}+\mathrm{c})\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}\right\}$$

$$=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\left\{\sqrt{l}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}\right\},=0\,,$$

viz., $\sqrt{l_1}\mp(\sqrt{m_1}+\sqrt{n_1})$ with a properly selected signification of the sign $\mp$ is $=0$; and similarly $\sqrt{l_2}\mp(\sqrt{m_2}+\sqrt{n_2})$ with a properly selected signification of the sign $\mp$ is $=0$; that is, each of the trizomals is a cubic.

199. IV. $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$ (values which, be it observed, satisfy of themselves the above assumed equation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$); the order of the tetrazomal is $=6$; and the order of each of the trizomals is here again $=3$. We in fact have $\sqrt{l_1}=\mathrm{a}+\mathrm{d}$, $\sqrt{m_1}+\sqrt{n_1}=\mathrm{b}+\mathrm{c}$, and therefore $\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=0$; and similarly $\sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2}=0$; that is, each of the trizomals is a cubic.

I attend, in particular, to the case where the four circles reduces themselves to the points *A*, *B*, *C*, *D*; these four points are then in a circle; and the curve under consideration is

$$\mathrm{a}\sqrt{\mathsf{A}}+\mathrm{b}\sqrt{\mathsf{B}}+\mathrm{c}\sqrt{\mathsf{C}}\,\mathrm{d}\sqrt{\mathsf{D}}=0\,;$$

in the general case where the points *A*, *B*, *C*, *D* are not on a circle, this is, as has been seen, a sextic curve, the locus of the foci of the conics which pass through the four given points; in the case where the points are in a circle then the sextic breaks up into two cubics (viz., observing that the curve under consideration is $\sqrt{l\mathsf{A}}+\sqrt{m\mathsf{B}}+\sqrt{n\mathsf{C}}+\sqrt{p\mathsf{D}}=0$, where $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$, these values do of themselves satisfy the condition of decomposability $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$), that is, the locus of the foci of the conics which pass through four points on a circle is composed of two circular cubics, each of them having the four points for a set of concyclic foci. It is easy to see why the sextic, thus defined as a locus of foci, must break up into two cubics; in fact, as we have seen, the conics which pass through the four concyclic points *A*, *B*, *C*, *D* have their axes in two fixed directions; there is consequently a locus of the foci situate on the axes which are in one of the fixed directions, and a separate locus of the foci situate on the axes which lie in the other of the fixed directions; viz., each of these loci is a circular cubic.

200. Adopting the notation of No. 188, or writing

$$RA = a_1,\ RB = b_1,\ RC = c_1,\ RD = d_1\ ,$$

(and therefore $b_1c_1 = a_1d_1$) we have

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = -\, d_1(b_1-c_1) : c_1(a_1-d_1) : -b_1(a_1-d_1) : a_1(b_1-c_1) .$$

Moreover

$$\begin{aligned} \sqrt{l_1} &= \mathrm{a} + \mathrm{d} &&, & \sqrt{l_2} &= \mathrm{a} + \mathrm{d}\,, \\ \sqrt{m_1} &= \mathrm{b} + \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}} &&, & \sqrt{m_2} &= \mathrm{b} - \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}\,, \\ \sqrt{n_2} &= \mathrm{c} - \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}} &&, & \sqrt{n_2} &= \mathrm{c} + \sqrt{\frac{\mathrm{bcb}}{\mathrm{a}}}\,, \end{aligned}$$

and we have

$$\frac{\mathrm{bcd}}{\mathrm{a}} = (a_1-d_1)^2\,\frac{\mathrm{a}_1\mathrm{b}_1\mathrm{c}_1}{d_1} = a_1{}^2(a_1-d_1)^2\,,\ \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}} = -\,a_1(a_1-d_1) \quad \textbf{suppose};$$

and thence

$$\begin{aligned} \sqrt{l_1} &= (a_1-d_1)(b_1-c_1)\,, & \sqrt{l_2} &= (a_1-d_1)(\ \ b_1-c_1) \\ \sqrt{m_1} &= (a_1-d_1)(c_1-a_1)\,, & \sqrt{m_2} &= (a_1-d_1)(\ \ c_1+a_1) \\ \sqrt{n_1} &= (a_1-d_1)(a_1-b_1)\,, & \sqrt{n_2} &= (a_1-d_1)(-a_1-b_1)\,, \end{aligned}$$

that is

$$\begin{aligned} \sqrt{l_1} : \sqrt{m_1} : \sqrt{n_1} &= b_1-c_1 : c_1-a_1 : \quad a_1-b_1\,, \\ \sqrt{l_2} : \sqrt{m_2} : \sqrt{n_2} &= b_1-c_1 : c_1+a_1 : -\,a_1-b_1\,, \end{aligned}$$

agreeing with the formulæ No. 188.

The tetrazomal curve

$$-d_1(b_1-c_1)\sqrt{\mathsf{A}} + c_1(a_1-d_1)\sqrt{\mathsf{B}} - b_1(a_1-d_1)\sqrt{\mathsf{C}} + a_1(b_1-c_1)\sqrt{\mathsf{D}} = 0$$

is thus decomposed into the two trizomals

$$\begin{aligned} (b_1-c_1)\sqrt{\mathsf{A}} + (c_1-a_1)\sqrt{\mathsf{B}} + (a_1-b_1)\sqrt{\mathsf{C}} &= 0\,, \\ (b_1-c_1)\sqrt{\mathsf{A}} + (c_1+a_1)\sqrt{\mathsf{B}} - (a_1+b_1)\sqrt{\mathsf{C}} &= 0\,. \end{aligned}$$

201. Observe that the tetrazomal equation is a consequence of either of the trizomal equations; taking for instance the first trizomal equation, this gives the tetrazomal equation, and consequently any combination of the trizomal equation and the tetrazomal equation is satisfied if only the trizomal equation is satisfied.

Multiply the trizomal equation by $-a_1+d_1$ and add it to the tetrazomal equation; the resulting equation contains the factor a_1, and omitting this, it is

$$(b_1-c_1)(-\sqrt{\mathsf{A}}+\sqrt{\mathsf{D}})+(a_1-d_1)(\sqrt{\mathsf{B}}-\sqrt{\mathsf{C}})=0,$$

where observe that b_1-c_1 is the distance BC, and a_1-d_1 the distance AD. But in like manner multiplying the second trizomal equation by $-a_1+d_1$, and adding it to the original tetrazomal equation, the resulting equation, omitting the factor a_1, is

$$(b_1-c_1)(-\sqrt{\mathsf{A}}+\sqrt{\mathsf{D}})-(a_1-d_1)(\sqrt{\mathsf{B}}-\sqrt{\mathsf{C}})=0;$$

viz., it is in fact the same tetrazomal equation as was obtained by means of the first trizomal equation.

201. The new tetrazomal equation, say

$$(b_1-c_1)(-\sqrt{\mathsf{A}}+\sqrt{\mathsf{D}})+(a_1-d_1)(\sqrt{\mathsf{B}}-\sqrt{\mathsf{C}})=0,$$

is thus equivalent to the original tetrazomal equation; observe that it is an equation of the form $\sqrt{l\mathsf{A}}+\sqrt{m\mathsf{B}}+\sqrt{n\mathsf{C}}+\sqrt{p\mathsf{D}}=0$, where

$$\sqrt{l}=-(b_1-c_1),\ \sqrt{m}=a_1-d_1,\ \sqrt{n}=(a_1-d_1),\ \sqrt{p}=b_1-c_1,$$

and where consequently $\sqrt{l}+\sqrt{p}=0$, $\sqrt{m}+\sqrt{n}=0$, that is an equation of the form (198) III., decomposable, as it should be, into the equations of two circular cubics. Writing

$$\frac{-\sqrt{\mathsf{A}}+\sqrt{\mathsf{D}}}{a_1-d_1}=\theta,\ \frac{\sqrt{\mathsf{B}}-\sqrt{\mathsf{C}}}{b_1-c_1}=\theta;$$

where θ is an arbitrary parameter, the curve is obtained as the locus of the intersections of two similar conics having respectively the foci (A, D) and the foci (B, C); (see SALMON, *Higher Plane Curves*, p. 174): whence we have the theorem, that if A, B, C, D are any four points on a circle, the two circular cubics which are the locus of the foci of the conics which pass through the four points A, B, C, D, are also the locus of the intersections of the similar conics, which have for their foci (A, D) and (B, C) respectively; and of the similar conics with the foci (B, D) and (C, A) respectively; and of the similar conics with the foci (C, D) and (A, B) respectively.

202. V. $\sqrt{l}=\sqrt{m}=\sqrt{n}=\sqrt{p}$. The order of the tetrazomal is $=5$, whence those of the trizomals should be $=3$ and $=2$ respectively. To verify this observe that the equation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$ gives $\frac{1}{\mathrm{a}}+\frac{1}{\mathrm{b}}+\frac{1}{\mathrm{c}}+\frac{1}{\mathrm{d}}=0$, and combining with $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0$, these are only satisfied by one of the systems

$(a + b = 0,\ c + d = 0)$, $(a + c = 0,\ b + d = 0)$, $(a + d = 0,\ b + c = 0)$. Selecting to fix the ideas the first of these, or writing

$$(a, b, c, d) = (a, -a, c, -c),$$

so that we have identically

$$a(A^\circ - B^\circ) + c(C^\circ - D^\circ) = 0,$$

an equation which signifies that the radical axis of the circles A, B is also the radical axis of the circles C, D; then, writing as we may do, $\sqrt{\frac{a}{bcd}\frac{p}{l}}\left(=\sqrt{\frac{1}{c^2}}\right)=\frac{1}{c}$, we have

$$\sqrt{l_1} = 1 - \frac{a}{c}, \qquad \sqrt{l_2} = 1 - \frac{a}{c},$$

$$\sqrt{m_1} = 1 + \frac{a}{c}, \qquad \sqrt{m_2} = 1 - \frac{a}{c},$$

$$\sqrt{n_1} = 1 + 1,\ = 2, \quad \sqrt{n_2} = 1 - 1,\ = 0.$$

Here $\sqrt{l_1} + \sqrt{m_1} - \sqrt{n_1} = 0$, which gives one of the trizomals a cubic, viz., this is the trizomal

$$\left(1 - \frac{a}{c}\right)\sqrt{A^\circ} + \left(1 + \frac{a}{c}\right)\sqrt{B^\circ} + 2\sqrt{C^\circ} = 0.$$

The other trizomal reduces itself to the bizomal $\sqrt{A^\circ} + \sqrt{B^\circ} = 0$, which regarded as a trizomal, or written under the form $(\sqrt{A^\circ} + \sqrt{B^\circ})^2 = 0$, is the line $A^\circ - B^\circ = 0$ twice, viz., this is the radical axis of the circles A_1, B_1 twice; and the order is thus $=2$. By what precedes, the line in question is in fact the common radical axis of the circles A, B and of the circles C, D.

Cases of the Decomposable Curve, the Centres in a Line—Art. Nos. 203 to 206.

203. We have yet to consider the decomposable case when the centres A, B, C, D are on a line; the equation $aA^\circ + bB^\circ + cC^\circ + dD^\circ = 0$ here subsists universally, whatever be the radii a'', b'', c'', d''. We establish as before the relation $\frac{l}{a} + \frac{m}{b} + \frac{n}{c} + \frac{p}{d} = 0$. The cases are as follows:—

I. No further relation between l, m, n, p, order of tetrazomal $= 8$, of trizomals 4 and 4.

II. $\sqrt{l} + \sqrt{m} + \sqrt{n} + \sqrt{p} = 0$; order of tetrazomal $= 7$; of trizomals $= 4$ and 3; same as II. *supra*

III. $\sqrt{l} + \sqrt{p} = 0$, $\sqrt{m} + \sqrt{n} = 0$; order of tetrazomal $= 6$; of trizomals 3 and 3; same as III. *supra*.

204. IV. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$, $\mathrm{a}\sqrt{l}+\mathrm{b}\sqrt{m}+\mathrm{c}\sqrt{n}+\mathrm{d}\sqrt{p}=0$; order of tetrazomal $=6$; this is a remarkable case, the orders of the trizomals are either 3, 3 or else 4, 2.

To explain how this is, it is to be noticed that in the absence of any special relation between the radii, the above conditions combined with $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$ give $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$*; when l, m, n, p have these values, the case is the same as IV. *supra*, and the orders of the trizomals are 3, 3. But if the radii of the circles satisfy the condition

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a''^2 & b''^2 & c''^2 & d''^2 \end{vmatrix}=0,$$

then the two conditions satisfy of themselves the remaining condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$, and the ratios $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}$ instead of being determinate as above, depend on an arbitrary parameter.

We have

$$\sqrt{l_1}=\sqrt{l}+\frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}},\quad \sqrt{m_1}=\sqrt{m}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{b}\sqrt{n},\quad \sqrt{n_1}=\sqrt{n}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{c}\sqrt{m},$$

and between l, m, n, p only the relations

$$\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0,\quad a\sqrt{m}+b\sqrt{m}+c\sqrt{n}+d\sqrt{p}=0.$$

We find first

$$\begin{aligned}\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1} &= \sqrt{l}+\sqrt{m}+\sqrt{n} \\ &\quad+\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{\mathrm{a}}{\mathrm{d}}\sqrt{p}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}(\mathrm{b}\sqrt{n}-\mathrm{c}\sqrt{m})\right\} \\ &=-\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{1}{\mathrm{d}}(\mathrm{d}\sqrt{l}-\mathrm{a}\sqrt{p})-\sqrt{\frac{\mathrm{a}}{\mathrm{bc}}}(\mathrm{b}\sqrt{n}-\mathrm{c}\sqrt{m})\right\},\end{aligned}$$

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* Writing x^2, y^2, z^2, w^2 in place of $\sqrt{l}, \sqrt{m}, \sqrt{n}, \sqrt{p}$, we have to find x, y, z, w from the conditions

$$\begin{aligned} x+y+z+w&=0,\\ ax+by+cz+dw&=0,\\ \frac{x^2}{\mathrm{a}}+\frac{y^2}{\mathrm{b}}+\frac{z^2}{\mathrm{c}}+\frac{w^2}{\mathrm{d}}&=0,\end{aligned}$$

where the constants are connected by the relation

$$a\mathrm{a}+b\mathrm{b}+c\mathrm{c}+d\mathrm{d}=0.$$

It readily appears that the line represented by the first two equations *touches* the quadric surface in the point $x:y:z:w=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$, so that these are in general the only values of $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}$. In the case next referred to in the text the line lies in the surface, and the values are not determined.

and then

$$(d-a)\sqrt{l}=(b-d)\sqrt{m}+(c-d)\sqrt{n}\,,$$
$$(d-a)\sqrt{p}=(a-b)\sqrt{m}+(a-c)\sqrt{n}\,,$$

whence

$$d\sqrt{l}-a\sqrt{p}=\frac{b-c}{d-a}(\mathrm{b}\sqrt{n}-\ \sqrt{m})\,,$$

and we have thus

$$\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=\frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left(\frac{b-c}{d-a}-\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\right)(\mathrm{b}\sqrt{n}-\mathrm{c}\sqrt{m})\,.$$

And similarly

$$\sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2}=\frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left(\frac{b-c}{d-a}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\right)(\mathrm{b}\sqrt{n}-\mathrm{c}\sqrt{m}):$$

(observe that in the case not under consideration $\mathrm{b}\sqrt{n}-\mathrm{c}\sqrt{m}=0$, and therefore $\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=0$, $\sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2}=0$). In the present case we have

$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=(b-c)(c-d)(d-b):-(c-d)(d-a)(a-b):-(d-a)(a-b)(b-d):-(a-b)(b-c)(c-a)$,

and thence

$$\frac{\mathrm{ad}}{\mathrm{bc}}=\frac{(b-c)^2}{(d-a)^2}\,,$$

so that only one of the two sums $\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}$, $\sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2}$ is $=0$, viz., assuming

$$\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}=\frac{b-c}{d-a}\,,$$

we have

$$\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=0\,.$$

And then also

$$\begin{aligned}a\sqrt{l_1}+b\sqrt{m_1}+c\sqrt{n_1}&=a\sqrt{l}+b\sqrt{m}+c\sqrt{n}\\&\quad+\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{\mathrm{a}a\sqrt{p}}{\mathrm{d}}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,(b\mathrm{b}\sqrt{n}-c\mathrm{c}\sqrt{m})\right\}\\&=-\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{1}{\mathrm{d}}(d\mathrm{d}\sqrt{l}-a\mathrm{a}\sqrt{p})-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}(b\mathrm{b}\sqrt{n}-c\mathrm{c}\sqrt{m})\right\};\end{aligned}$$

but we find

$$d\mathrm{d}\sqrt{l}-a\mathrm{a}\sqrt{p}=\frac{b-c}{d-a}(b\mathrm{b}\sqrt{n}-c\mathrm{c}\sqrt{m})\,,$$

and thence

$$a\sqrt{l_1}+b\sqrt{m_1}+c\sqrt{n_1}=\frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left(\frac{b-c}{d-a}-\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\right)(b\mathrm{b}\sqrt{n}-c\mathrm{c}\sqrt{m}),=\ 0\,,$$

in virtue of $\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}=\frac{b-c}{d-a}$. Hence $\sqrt{l_1}:\sqrt{m_1}:\sqrt{n_1},=b-c:c-a:a-b$, or the corresponding trizomal is a conic, but the other trizomal is a quartic.

205. V. $\sqrt{l} = \sqrt{m} = \sqrt{n} = \sqrt{p}$; order of tetrazomal is $= 5$; orders of trizomals $= 3, 2$; same as V. *supra.*

VI. $\sqrt{l} + \sqrt{p} = 0$, $\sqrt{m} + \sqrt{n} = 0$, $a\sqrt{l} + b\sqrt{m} + c\sqrt{n} + d\sqrt{p} = 0$; order of tetrazomal $= 5$; orders of trizomals are 3, 2.

We have here

$$\sqrt{l_1} = \frac{\mathrm{a} + \mathrm{d}}{\mathrm{d}} \sqrt{l}\,,$$

$$\sqrt{m_1} = \sqrt{m} + \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\mathrm{b}\sqrt{m}\,,$$

$$\sqrt{n_1} = \sqrt{m} + \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\mathrm{c}\sqrt{m}\,,$$

or writing the values of $\sqrt{m_1}$, $\sqrt{n_1}$ in the form

$$\sqrt{m_1} = \sqrt{m} + \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\,\frac{\mathrm{b}}{\mathrm{d}}\sqrt{m}\,,$$

$$\sqrt{n_1} = -\sqrt{m} + \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\,\frac{\mathrm{c}}{\mathrm{d}}\sqrt{m}\,,$$

then observing that as before $l = \frac{\mathrm{ad}}{\mathrm{bc}} m$, if to fix the ideas we assume $\sqrt{l} = \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}$, the equations are

$$\sqrt{l_1} = \frac{\mathrm{a} + \mathrm{d}}{\mathrm{d}}\sqrt{l} \qquad \text{and similarly } \sqrt{l_2} = \frac{\mathrm{a} + \mathrm{d}}{\mathrm{d}}\sqrt{l}$$

$$\sqrt{m_1} = \sqrt{m} + \frac{\mathrm{b}}{\mathrm{d}}\sqrt{l}\,, \qquad \sqrt{m_2} = \sqrt{m} - \frac{\mathrm{b}}{\mathrm{d}}\sqrt{l}\,,$$

$$\sqrt{n_1} = -\sqrt{m} + \frac{\mathrm{c}}{\mathrm{d}}\sqrt{l}\,, \qquad \sqrt{n_2} = \sqrt{m} - \frac{\mathrm{c}}{\mathrm{d}}\sqrt{l}\,,$$

whence

$$\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = 0\,,\ \sqrt{l_2} - \sqrt{m_2} - \sqrt{n_2} = 0\,.$$

We have moreover

$$a\sqrt{l_1} = \frac{a\mathrm{a} + a\mathrm{d}}{\mathrm{d}}\sqrt{l}\,,$$

$$b\sqrt{m_1} + c\sqrt{n_1} = (b - c)\sqrt{m} + \frac{b\mathrm{b} + c\mathrm{c}}{\mathrm{d}}\sqrt{l}\,,$$

and thence

$$a\sqrt{l_1} + b\sqrt{m_1} + c\sqrt{n_1} = (a - d)\sqrt{l} + (b - c)\sqrt{m} = 0\,,$$

so that

$$\sqrt{l_1} : \sqrt{m_1} : \sqrt{n_1} = b - c : c - a : a - b\,;$$

the corresponding trizomal is thus a conic, and it has been seen that the other trizomal is a cubic.

VII. If we have $\begin{vmatrix} 1, & 1, & 1, & 1 \\ a, & b, & c, & d \\ a^2, & b^2, & c^2, & d^2 \\ a''^2, & b''^2, & c''^2, & d''^2 \end{vmatrix} = 0$, and $\left(\begin{array}{cccc} 1, & 1, & 1, & 1 \\ a, & b, & c, & d \\ a^2, & b^2, & c^2, & d^2 \\ a''^2, & b''^2, & c''^2, & d''^2 \end{array}\right)\left(\sqrt{l}, \sqrt{m}, \sqrt{n}, \sqrt{p}\right) = 0$,

the tetrazomal has a branch ideally containing ($z^3 = 0$) the line infinity 3 times; order is $= 5$; orders of the trizomals are 3, 2. We have here

$$\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = \mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d},$$

and thence

$$\sqrt{l_1} = \mathrm{a} + \mathrm{d} \quad , \quad \sqrt{l_2} = \mathrm{a} + \mathrm{d}$$

$$\sqrt{m_1} = \mathrm{b} - \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}, \quad \sqrt{m_2} = \mathrm{b} + \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}$$

$$\sqrt{n_1} = \mathrm{c} + \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}, \quad \sqrt{n_2} = \mathrm{c} - \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},$$

which give

$$\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = 0, \; \sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2} = 0.$$

Moreover

$$\begin{aligned} a\sqrt{l_1} + b\sqrt{m_1} + c\sqrt{n_1} &= a(\mathrm{a}+\mathrm{d}) + b\mathrm{b} + c\mathrm{c} \\ &\quad - (b-c)\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}} \\ &= (a-d)\mathrm{d} - (b-c)\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}} \\ &= \mathrm{d}\left\{(a-d) - (b-c)\sqrt{\frac{\mathrm{bc}}{\mathrm{ad}}}\right\}, \end{aligned}$$

and similarly

$$a\sqrt{l_2} + b\sqrt{m_2} + c\sqrt{n_2} = \mathrm{d}\left\{(a-d) + (b-c)\sqrt{\frac{\mathrm{bc}}{\mathrm{ad}}}\right\};$$

whence in virtue of

$$\frac{\mathrm{ad}}{\mathrm{bc}} = \frac{(b-c)^2}{(d-a)^2},$$

one of the two expressions is $= 0$; and the trizomals are thus a conic and a cubic.

The Decomposable Curve; Transformation to a different set of Concyclic Foci—Art. No. 206.

206. Consider the decomposable case of

$$\sqrt{l}\mathbf{A} + \sqrt{m}\mathbf{B} + \sqrt{n}\mathbf{C} + \sqrt{p}\mathbf{D} = 0;$$

viz., the points A, B, C, D lie here in a circle, and we have $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0$. Taking (A_1, D_1) the anti-points of (A, D); (B_1, C_1) the anti-points of (B, C); then $\mathsf{A}_1\mathsf{D}_1 = \mathsf{AD}$, $\mathsf{B}_1\mathsf{C}_1 = \mathsf{BC}$ (No. 65) and referring to the formulæ, *ante*, Nos. 100 *et seq.*, it appears that we can find l_1, m_1, n_1, p_1 such that identically

$$- l\mathsf{A} + m\mathsf{B} + n\mathsf{C} - p\mathsf{D} = - l_1\mathsf{A}_1 + m_1\mathsf{B}_1 + n_1\mathsf{C}_1 - p_1\mathsf{D}_1 ,$$

and moreover that $lp = l_1 p_1$, $mn = m_1 n_1$.
The equation of the curve gives

$$- l\mathsf{A} + m\mathsf{B} + n\mathsf{C} - p\mathsf{D} - 2\sqrt{lp\mathsf{AD}} + 2\sqrt{mn\mathsf{BC}} = 0 ,$$

which may consequently be written

$$- l_1\mathsf{A}_1 + m_1\mathsf{B}_1 + n_1\mathsf{C}_1 - p_1\mathsf{D}_1 - 2\sqrt{l_1p_1\mathsf{A}_1\mathsf{D}_1} + 2\sqrt{m_1n_1\mathsf{B}_1\mathsf{C}_1} = 0 ;$$

viz., this is

$$\sqrt{l_1\mathsf{A}_1} + \sqrt{m_1\mathsf{B}_1} + \sqrt{n_1\mathsf{C}_1} + \sqrt{p_1\mathsf{D}_1} = 0 ;$$

that is, the two trizomals expressed by the original tetrazomal equation involving the set of concyclic foci (A, B, C, D) are thus expressed by a new tetrazomal equation involving the different set of concyclic foci (A_1, B_1, C_1, D_1); and we might of course in like manner express the equation in terms of the other two sets of concyclic foci (A_2, B_2, C_2, D_2) and (A_3, B_3, C_3, D_3) respectively. It might have been anticipated that such a transformation existed, for we could as regards each of the component trizomals separately pass from the original set to a different set of concyclic foci, and the two trizomal equations thus obtained would, it might be presumed, be capable of composition into a single tetrazomal equation; but the direct transformation of the tetrazomal equation is not on this account less interesting.

ANNEX I.—*On the Theory of the Jacobian.*

Consider any three curves $U = 0$, $V = 0$, $W = 0$, of the same order r, then writing

$$J(U, V, W) = \frac{d(U, V, W)}{d(x, y, z)} = \begin{vmatrix} d_xU, & d_xV, & d_xW \\ d_yU, & d_yV, & d_yW \\ d_zU, & d_zV, & d_zW \end{vmatrix} ,$$

we have the Jacobian curve $J(U, V, W) = 0$, of the order $3r - 3$.

A fundamental property is that if the curves $U = 0$, $V = 0$, $W = 0$ have any common point, this is a point on the Jacobian, and not only so, but it is a node, or double point, that is, for the point in question we have $J = 0$, and also $d_xJ = 0$, $d_yJ = 0$, $d_zJ = 0$.

It follows that for the three curves $l\Theta + L\Phi = 0$, $m\Theta + M\Phi = 0$, $n\Theta + N\Phi = 0$ $(\Theta = 0$ of the order $r - s'$, $\Phi = 0$ of the order $r - s$, $l = 0$, $m = 0$, $n = 0$ each of the order s', $L=0$, $M=0$, $N=0$ each of the order s) which have in common the $(r-s')(r-s)$ points of intersection of the curves $\Theta=0$, $\Phi=0$, each of these points is a node on the Jacobian, and hence that the Jacobian must be of the form

$$J(l\Theta+L\Phi,\ m\Theta+M\Phi,\ n\Theta+N\Phi)=A\Theta^2+2B\Theta\Phi+C\Phi^2=0\,,$$

where obviously the degrees of A, B, C must be $r+2s'-3$, $r+s+s'-3$, $r+2s-3$ respectively. In the particular case where $s'=0$, that is where l, m, n are constants, we have $A=0$; the Jacobian curve then contains as a factor $(\Phi=0)$, and throwing this out, the curve is $B\Theta+C\Phi=0$, viz., this is a curve of the order $2r+s-3$ passing through each of $r(r-s)$ points of intersection of the curves $\Theta=0$, $\Phi=0$.

In particular, if $r=2$, $s=1$, that is, if the curves are the conics $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, $\Theta+N\Phi=0$, passing through the two points of intersection of the conic $\Theta=0$ by the line $\Phi=0$, then the Jacobian is a conic passing through these same two points, viz., its equation is of the form $\Theta+\Omega\Phi=0$. This intersects any one of the given conics, say $\Theta+L\Phi=0$ in the points $\Theta=0$, $\Phi=0$, and in two other points $\Theta+\Omega\Phi=0$, $\Omega-L=0$; at *each* of the last-mentioned points, the tangents to the two curves, and the lines drawn to the two points $\Theta=0$, $\Phi=0$, form a harmonic pencil.

Although this is, in fact, the known theorem that the Jacobian of three circles is their orthotomic circle, yet it is, I think, worth while to give a demonstration of the theorem as above stated in reference to the conics through two given points.

Taking $(z=0,\ x=0)(z=0,\ y=0)$ for the two given points $\Theta=0$, $\Phi=0$, the general equation of a conic through the two points is a quadric equation containing terms in z^2, zx, zy, xy; taking any two such conics

$$\begin{aligned} cz^2+2fyz+2gzx+2hxy&=0\,,\\ Cz^2+2Fyz+2Gzx+2Hxy&=0\,,\end{aligned}$$

these intersect in the two points $(x=0,\ z=0)$, $(y=0,\ z=0)$ and in two other points; let (x,y,z) be co-ordinates of either of the last-mentioned points, and take (X,Y,Z) as current co-ordinates, the equations of the lines to the fixed points and of the two tangents are

$$\begin{gathered} Xz-Zx=0\,,\ Yz-Zy=0\,,\\ (hy+gz)(Xz-Zx)+(hx+fz)(Yz-Zy)=0\,,\\ (Hy+Gz)(Xz-Zx)+(Hx+Fz)(Yz-Zy)=0\,,\end{gathered}$$

whence the condition for the harmonic relation is

$$(hy+gz)(Hx+Fz)+(hx+fz)(Hy+Gz)=0\,,$$

that is

$$(fG+gF)z^2+(hF+fH)yz+(gH+hG)zx+2hHxy=0\,,$$

but from the equations of the two conics multiplying by $\tfrac{1}{2}H$, $\tfrac{1}{2}h$ and adding, we have

$$\tfrac{1}{2}(cH + hC)z^2 + (hF + fH)yz + (gH + hG)zx + 2hHxy = 0\,;$$

viz., the condition is thus reduced to

$$cH + hC - 2(fG + gF) = 0\,,$$

so that being satisfied for one of the points in question, it will be satisfied for the other of them. Now for the three conics

$$\begin{aligned} cz^2 &+ 2fyz + 2gzx + 2hxy = 0\,, \\ c'z^2 &+ 2f'yz + 2g'zx + 2h'xy = 0\,, \\ c''z^2 &+ 2f''yz + 2g''zx + 2h''xy = 0\,, \end{aligned}$$

forming the Jacobian, and throwing out the factor z, we may write the equation in the form

$$Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0\,,$$

where the values are

$$\begin{aligned} C &= g(f'c'' - f''c') + g'(f''c - fc'') + g''(fc' - f'c)\,, \\ H &= g(h'f'' - h''f') + g'(h''f - hf'') + g''(hf' - h'f)\,, \\ 2F &= h(f'c'' - f''c') + h'(f''c - fc'') + h''(fc' - f'c)\,, \\ 2G &= h(c'g'' - c''g') + h'(c''g - cg'') + h''(cg' - c'g)\,; \end{aligned}$$

and we thence obtain

$$\begin{aligned} cH + hC &= -(fg' - f'g)(c''h - ch'') + (f''g - fg'')(ch' - c'h) \\ &= 2(fG + gF)\,, \end{aligned}$$

viz., the condition is satisfied in regard to the Jacobian and the first of the three conics; and it is therefore also satisfied in regard to the Jacobian and the other two conics respectively.

I do not know any general theorem in regard to the Jacobian which gives the foregoing theorem of the orthotomic circle. It may be remarked that the use in the Memoir of the theorem of the orthotomic circle is not so great as would at first sight appear: it fixes the ideas to speak of the orthotomic circle of three given circles rather than of their Jacobian, but we are concerned with the orthotomic circle less as the circle which cuts at right angles the given circles than as a circle standing in a known relation to the given circles.

ANNEX II.—*On* CASEY'S *Theorem for the Circle which touches three given Circles.*

The following two problems are identical:—

1. To find a circle touching three given circles.

2. To find a cone-sphere (sphere the radius of which is = 0) passing through three given points in space.

In fact, in the first problem if we use z to denote a given constant (which may be $= 0$), then taking a, a' and $i(z - a'')$ for the co-ordinates of the centre and for the radius of one of the given circles; and similarly $b, b', i(z - b'')$; $c, c', i(z - c'')$ for the other two given circles; and $S, S', i(z - S'')$ for the required circle; the equations of the given circles will be

$$\begin{aligned}(x - a)^2 + (y - a')^2 + (z - a'')^2 = 0\,,\\ (x - b)^2 + (y - b')^2 + (z - b'')^2 = 0\,,\\ (x - c)^2 + (y - c')^2 + (z - c'')^2 = 0\,,\end{aligned}$$

and that of the required circle will be

$$(x-S)^2 + (y-S')^2 + (z-S''')^2 = 0\,.$$

In order that this may touch the given circles, the distances of its centre from the centres of the given circles must be $i(S''-a'')$, $i(S''-b'')$, $i(S''-c'')$ respectively; the conditions of contact then are

$$\begin{aligned}(S-a)^2 + (S'-a')^2 + (S'''-a'')^2 = 0\,,\\ (S-b)^2 + (S'-b')^2 + (S'''-b'')^2 = 0\,,\\ (S-c)^2 + (S'-c')^2 + (S'''-c'')^2 = 0\,,\end{aligned}$$

or we have from these equations to determine S, S', S''. But taking (a, a', a''), (b, b', b''), (c, c', c'') for the co-ordinates of three given points in space, and (S, S', S'') for the co-ordinates of the centre of the cone-sphere through these points, we have the very same equations for the determination of (S, S', S''), and the identity of the two problems thus appears.

I will presently give the direct analytical solution of this system of equations. But to obtain a solution in the form required, I remark that the equation of the cone-sphere in question is nothing else than the relation that exists between the co-ordinates of any four points on a cone-sphere; to find this, consider any five points in space, 1, 2, 3, 4, 5; and let $\overline{12}$, &c. denote the distances between the points 1 and 2, &c.; then we have between the distances of the five points the relation

$$\begin{vmatrix} 0, & 1\;, & 1\;, & 1\;, & 1\;, & 1 \\ 1, & 0\;, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2 \\ 1, & \overline{21}^2, & 0\;, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2 \\ 1, & \overline{31}^2, & \overline{32}^2, & 0\;, & \overline{34}^2, & \overline{35}^2 \\ 1, & \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0\;, & \overline{45}^2 \\ 1, & \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0 \end{vmatrix} = 0\,;$$

whence taking 5 to be the centre of the cone-sphere through the points 1, 2, 3, 4, we have $\overline{15} = \overline{25} = \overline{35} = \overline{45} = 0$; and the equation becomes

$$\begin{vmatrix} 0 \;, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ \overline{21}^2, & 0 \;, & \overline{23}^2, & \overline{24}^2 \\ \overline{31}^2, & \overline{32}^2, & 0 \;, & \overline{34}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = 0,$$

which is the relation between the distances of any four points on a cone-sphere; this equation may be written under the irrational form

$$\overline{23}\,.\,\overline{14} + \overline{31}\,.\,\overline{24} + \overline{12}\,.\,\overline{34} = 0.$$

Taking (a, a', a''), (b, b', b''), (c, c', c''), (x, y, z) for the co-ordinates of the four points respectively, we have

$$\begin{aligned} \overline{23} &= \sqrt{(b-c)^2 + (b'-c')^2 + (b''-c'')^2}, & \overline{14} &= \sqrt{(x-a)^2 + (y-a')^2 + (z-a'')^2}, \\ \overline{31} &= \sqrt{(c-a)^2 + (c'-a')^2 + (c''-a'')^2}, & \overline{24} &= \sqrt{(x-b)^2 + (y-b')^2 + (z-b'')^2}, \\ \overline{12} &= \sqrt{(a-b)^2 + (a'-b')^2 + (a''-b'')^2}, & \overline{34} &= \sqrt{(x-c)^2 + (y-c')^2 + (z-c'')^2}, \end{aligned}$$

or the symbols having these significations, we have

$$\overline{23}\,.\,\overline{14} + \overline{31}\,.\,\overline{24} + \overline{12}\,.\,\overline{34} = 0$$

for the equation of the cone-sphere through the three points; or rather (since the rational equation is of the order 4 in the co-ordinates (x, y, z)) this is the equation of the pair of cone-spheres through the three given points; and similarly it is in the first problem the equation of a pair of circles each touching the three given circles respectively.

In the first problem the radii of the given circles were $i(z-a'')$, $i(z-b'')$, $i(z-c'')$ respectively; denoting these radii by α, β, γ, or taking the equations of the given circles to be

$$\begin{aligned} (x-a)^2 + (y-a')^2 - \alpha^2 &= 0, \\ (x-b)^2 + (y-b')^2 - \beta^2 &= 0, \\ (x-c)^2 + (y-c')^2 - \gamma^2 &= 0, \end{aligned}$$

the symbols then are

$$\begin{aligned} \overline{23} &= \sqrt{(b-c)^2 + (b'-c')^2 - (\beta-\gamma)^2}, & \overline{14} &= \sqrt{(x-a)^2 + (y-a')^2 - \alpha^2}, \\ \overline{31} &= \sqrt{(c-a)^2 + (c'-a')^2 - (\gamma-\alpha)^2}, & \overline{24} &= \sqrt{(x-b)^2 + (y-b')^2 - \beta^2}, \\ \overline{12} &= \sqrt{(a-b)^2 + (a'-b')^2 - (\alpha-\beta)^2}, & \overline{34} &= \sqrt{(x-c)^2 + (y-c')^2 - \gamma^2}, \end{aligned}$$

and the equation of the pair of circles is as before

$$\overline{23}\,.\,\overline{14} + \overline{31}\,.\,\overline{24} + \overline{12}\,.\,\overline{34} = 0;$$

where it is to be noticed that $\overline{23}$, $\overline{31}$, $\overline{12}$ are the tangential distances of the circles 2 and 3, 3 and 1, 1 and 2 respectively; viz., if α, β, γ are the radii taken positively, then these are the direct tangential distances. By taking the radii positively or negatively at pleasure, we obtain in all four equations—the tangential distances being all direct as above, or else any one is direct, and the other two are inverse; we have thus the four pairs of tangent circles.

The cone-spheres which pass through a given circle are the two spheres which have their centres in the two anti-points of the given circle; and it is easy to see that the foregoing investigation gives the following (imaginary) construction of the tangent circles; viz., given any three circles A, B, C in the same plane, to draw the tangent circles. Taking the anti-points of the three circles, then selecting any three anti-points (one for each circle) so as to form a triad, we have in all four complementary pairs of triads. Through a triad, and through the complementary triad draw two circles, these are situate symmetrically on opposite sides of the plane; and combining each anti-point of the first circle with the symmetrically situated anti-point of the second circle, we have two pairs of points, the points of each pair being symmetrically situate in regard to the plane, and having therefore an anti-circle in this plane; these two anti-circles are a pair of tangent circles; and the four pairs of complementary triads give in this manner the four pairs of tangent circles.

I return to the equations

$$\begin{aligned}
(x-S)^2+(y-S')^2+(z-S'')^2&=0,\\
(a-S)^2+(a'-S')^2+(a''-S'')^2&=0,\\
(b-S)^2+(b'-S')^2+(b''-S'')^2&=0,\\
(c-S)^2+(c'-S')^2+(c''-S'')^2&=0;
\end{aligned}$$

by eliminating (S, S', S'') from these equations we shall obtain the equation of the pair of cone-spheres through the points (a, a', a''), (b, b', b''), (c, c', c''). Write $x-S,\ y-S',\ z-S''=X,\ Y,\ Z$, then we have

$$X^2+Y^2+Z^2=0\,,$$

and if, for shortness, we put

$$\begin{aligned}
\mathsf{A}&=(a-x)^2+(a'-y)^2+(a''-z)^2\,,\\
\mathsf{B}&=(b-x)^2+(b'-y)^2+(b''-z)^2\,,\\
\mathsf{C}&=(c-x)^2+(c'-y)^2+(c''-z)^2\,,
\end{aligned}$$

then by means of the equation just obtained the other three equations become

$$\begin{aligned}
\mathsf{A}+2\big[(a-x)X+(a'-y)Y+(a''-z)Z\big]&=0\,,\\
\mathsf{B}+2\big[(b-x)X+(b'-y)Y+(b''-z)Z\big]&=0\,,\\
\mathsf{C}+2\big[(c-x)X+(c'-y)Y+(c''-z)Z\big]&=0\,.
\end{aligned}$$

These last equations give

$$\begin{aligned} X : Y : Z = {} & \lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C} \\ & : \lambda'\mathbf{A} + \mu'\mathbf{B} + \nu'\mathbf{C} \\ & : \lambda''\mathbf{A} + \mu''\mathbf{B} + \nu''\mathbf{C} \,, \end{aligned}$$

where

$$\begin{aligned} \lambda &= b'c'' - b''c' + (c' - b')z - (c'' - b'')y \,, \\ \mu &= c'a'' - c''a' + (a' - c')z - (a'' - c'')y \,, \\ \nu &= a'b'' - a''b' + (b' - a')z - (b'' - a'')y \,, \\ \lambda' &= b''c - bc'' + (c'' - b'')x - (c - b)z \,, \\ \mu' &= c''a - ca'' + (a'' - c'')x - (a - c)z \,, \\ \nu' &= a''b - ab'' + (b'' - a'')x - (b - a)z \,, \\ \lambda'' &= bc' - b'c + (c - b)y - (c' - b')x \,, \\ \mu'' &= ca' - c'a + (a - c)y - (a' - c')x \,, \\ \nu'' &= ab' - a'b + (b - a)y - (b' - a')x \,; \end{aligned}$$

and the result of the elimination then is

$$\begin{aligned} & (\lambda\,\mathbf{A} + \mu\mathbf{B} + \nu\,\mathbf{C})^2 \\ + & (\lambda'\,\mathbf{A} + \mu'\mathbf{B} + \nu'\mathbf{C})^2 \\ + & (\lambda''\mathbf{A} + \mu''\mathbf{B} + \nu''\mathbf{C})^2 = 0 \,. \end{aligned}$$

But substituting for **A**, **B**, **C** their values, and writing, for shortness,

$$\begin{aligned} -i &= b'c'' - b''c' + c'a'' - c''a' + a'b'' - a''b \,, \\ -j &= b''c - bc'' + c''a - ca'' + a''b - ab'' \,, \\ -k &= bc' - b'c + ca' - c'a + ab' - a'b \,, \\ \Delta &= a(b'c'' - b''c) + a'(b''c - bc'') + a''(bc' - b'c) \,, \\ -p &= (b'c'' - b''c')(a^2 + a'^2 + a''^2) + (c'a'' - c''a')(b^2 + b'^2 + b''^2) + (a'b'' - a''b')(c^2 + c'^2 + c''^2) \,, \\ -q &= (b''c - bc'')(a^2 + a'^2 + a''^2) + (c''a - ca'')(b^2 + b'^2 + b''^2) + (a''b - ab'')(c^2 + c'^2 + c''^2) \,, \\ -r &= (bc' - b'c)(a^2 + a'^2 + a''^2) + (ca' - c'a)(b^2 + b'^2 + b''^2) + (ab' - ab')(c^2 + c'^2 + c''^2) \,, \\ -l &= (c - b)(a^2 + a'^2 + a''^2) + (a - c)(b^2 + b'^2 + b''^2) + (b - a)(c^2 + c'^2 + c''^2) \,, \\ -m &= (c' - b')(a^2 + a'^2 + a''^2) + (a' - c')(b^2 + b'^2 + b''^2) + (b' - a')(c^2 + c'^2 + c''^2) \,, \\ -n &= (c'' - b'')(a^2 + a'^2 + a''^2) + (a'' - c'')(b^2 + b'^2 + b''^2) + (b'' - a'')(c^2 + c'^2 + c''^2) \,, \end{aligned}$$

we find

$$\begin{aligned} & \lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C} \\ & \quad = - \; i(x^2 + y^2 + z^2) \\ & \qquad + 2i(x^2 + y^2 + z^2) - 2x(ix + jy + kz) - 2\Delta x + ny - mz - p \,, \end{aligned}$$

with similar expressions for $\lambda'\mathbf{A} + \mu'\mathbf{B} + \nu'\mathbf{C}$, $\lambda''\mathbf{A} + \mu''\mathbf{B} + \nu''\mathbf{C}$, and the result is

$$\begin{aligned} & \{i(x^2 + y^2 + z^2) - 2x(ix + jy + kz) - 2\Delta x + ny - mz - p\}^2 \\ + & \{j(x^2 + y^2 + z^2) - 2y(ix + jy + kz) - nx - 2\Delta y + lz - q\}^2 \\ + & \{k(x^2 + y^2 + z^2) - 2z(ix + jy + kz) + mx - ly - 2\Delta z - r\}^2 = 0 \,, \end{aligned}$$

viz., this is

$$\begin{aligned}&(x^2+y^2+z^2)^2(i^2+j^2+k^2)\\ &+(x^2+y^2+z^2)\{4\Delta(ix+jy+kz+2\big(i(ny-mz)+j(lz-nx)+k(mx-ly)\big)\\ &\qquad+4\Delta^2-2(ip+jq+kr)+(l^2+m^2+n^2)\}\\ &-(lx+my+nz)^2+4(ix+jy+kz)(px+qy+rz)\\ &+4\Delta(px+qy+rz)-2\big(p(ny-mz)+q(lz-nx)+r(mx-ly)\big)\\ &+p^2+q^2+r^2=0\,.\end{aligned}$$

viz., this is in the rational form the equation of the pair of cone-spheres. The function on the left hand side must, it is clear, be save to a numerical factor the norm of

$$\begin{aligned}&\sqrt{(b-c)^2+(b'-c')^2+(b''-c'')^2}\,.\,\sqrt{(x-a)^2+(y-a')^2+(z-a'')^2}\\ +&\sqrt{(c-a)^2+(c'-a')^2+(c''-a'')^2}\,.\,\sqrt{(x-b)^2+(y-b')^2+(z-b'')^2}\\ +&\sqrt{(a-b)^2+(a'-b')^2+(a''-b'')^2}\,.\,\sqrt{(x-c)^2+(y-c')^2+(z-c'')^2}\,,\end{aligned}$$

the numerical factor of the expression in question is in fact $=-4$, that is, the norm is

$$=-4(x^2+y^2+z^2)^2(i^2+j^2+k^2)+\text{\&c.}\,;$$

so that attending only to the highest powers in (x, y, z) we ought to have

$$\begin{aligned}&\text{Norm}\,\{\sqrt{(b-c)^2+(b'-c')^2+(b''-c'')^2}+\sqrt{(c-a)^2+(c'-a')^2+(c''-a'')^2}+\sqrt{(a-b)^2+(a'-b')^2+(a''-b'')^2}\}\\ &\quad=-4(i^2+j^2+k^2)\,.\end{aligned}$$

It is easy to see that the norm is in fact composed of the terms

$$\begin{aligned}&2(b'-c')^2\{\ \ (b-c)^2-(c-a)^2-(a-b)^2\}\,,\\ +&2(c'-a')^2\{-(b-c)^2+(c-a)^2-(a-b)^2\}\,,\\ +&2(a'-b')^2\{-(b-c)^2-(c-a)^2+(a-b)^2\}\,,\end{aligned}$$

and of the similar terms (a, b, c), (a'', b'', c'') and in (a', b', c'), (a'', b'', c''); the above written terms are $=-4$ into

$$\begin{aligned}&(b'-c')^2(a-b)(a-c)\\ +&(c'-a')^2(b-c)(b-a)\\ +&(a'-b')^2(c-a)(a-b)\,,\end{aligned}$$

which is

$$\begin{aligned}&=a'^2(b-c)^2+b'^2(c-a)+c'^2(a-b)^2\\ &\qquad+2b'c'(a-b)(c-a)+2c'a'(b-c)(a-b)+2a'b'(c-a)(b-c)\\ &=\{a'(b-c)+b'(c-a)+c'(a-b)\}^2\\ &=k^2\,;\end{aligned}$$

and the value of the norm is thus $=-4(i^2+j^2+k^2)$, as it should be.

ANNEX III.—*On the Norm of* $(b-c)\sqrt{\mathrm{A}^\circ}+(c-a)\sqrt{\mathrm{B}^\circ}+(a-b)\sqrt{\mathrm{C}^\circ}$, *when the Centres are in a Line.*

The norm of $\sqrt{U}+\sqrt{V}+\sqrt{W}$ is

$$=(1,1,1,-1,-1,-1)(U,V,W)^2,$$

whence that of $\sqrt{U+U'}+\sqrt{V+V'}+\sqrt{W+W'}$ is

$$\begin{aligned}&=\ (1,1,1,-1,-1,-1)(U,V,W)^2\\&+\ (1,1,1,-1,-1,-1)(U',V',W')^2\\&+2(1,1,1,-1,-1,-1)(U,V,W)(U',V',W'),\end{aligned}$$

where the last term is $=2$ into

$$\begin{aligned}&U'(\ \ U-V-W)\\+\ &V'(-U+V-W)\\+\ &W'(-U-V+W).\end{aligned}$$

And the norm of $\sqrt{U+U'+U''}+\sqrt{V+V'+V''}+\sqrt{W+W'+W''}$ is obviously composed in a similar manner.

Now, applying the formula to obtain the norm of

$$(b-c)\sqrt{a^2+\theta+\alpha}+(c-a)\sqrt{b^2+\theta+\beta}+(a-b)\sqrt{c^2+\theta+\gamma},$$

the expression contains six terms, two of which are at once seen to vanish; and writing for shortness (,,) in place of (1, 1, 1, — 1, — 1, — 1) the remaining terms are

$$\begin{aligned}&(\,,,\,)\big((b-c)^2\alpha,\ (c-a)^2\beta,\ (a-b)^2\gamma\,\big)^2\\+\,2\ &(\,,,\,)\big((b-c)^2\alpha,\ (c-a)^2\beta,\ (a-b)^2\gamma\,\big)\big((b-c)^2a^2,\ (c-a)^2b^2,\ (a-b)^2c^2\big)\\+\,2\theta&(\,,,\,)\big((b-c)^2\alpha,\ (c-a)^2\beta,\ (a-b)^2\gamma\,\big)\big((b-c)^2\ \ ,\ (c-a)^2\ \ ,\ (a-b)^2\ \ \big)\\+\,2\theta&(\,,,\,)\big((b-c)^2a^2,\ (c-a)^2b^2,\ (a-b)^2c^2\big)\big((b-c)^2\ \ ,\ (c-a)^2\ \ ,\ (a-b)^2\ \ \big);\end{aligned}$$

the first of these terms requires no reduction; the second, omitting the factor 2, is

$$\begin{aligned}&(b-c)^2\alpha\big[\ \ (b-c)^2a^2-(c-a)^2b^2-(a-b)^2c^2\big]\\+\ &(c-a)^2\beta\big[-(b-c)^2a^2+(c-a)^2b^2-(a-b)^2c^2\big]\\+\ &(a-b)^2\gamma\big[-(b-c)^2a^2-(c-a)^2b^2+(a-b)^2c^2\big];\end{aligned}$$

which is

$$=2(a-b)(b-c)(c-a)\big[bc\,(b-c)\,\alpha+ca\,(c-a)\,\beta+ab\,(a-b)\gamma\big].$$

Similarly the third term, omitting the factor 2θ, is

$$\begin{aligned}&(b-c)^2\alpha\big[\ \ (b-c)^2-(c-a)^2-(a-b)^2\big]\\+\ &(c-a)^2\beta\big[-(b-c)^2+(c-a)^2-(a-b)^2\big]\\+\ &(a-b)^2\gamma\big[-(b-c)^2-(c-a)^2+(a-b)^2\big],\end{aligned}$$

which is

$$= 2(a-b)(b-c)(c-a)\left[(b-c)\alpha + (c-a)\beta + (a-b)\gamma\right],$$

and for the last term, omitting the factor 2θ, this may be deduced therefrom by writing (a^2, b^2, c^2) in place of (α, β, γ), viz., it is

$$= -2(a-b)^2(b-c)^2(c-a)^2 .$$

Hence, restoring the omitted factors, and collecting, we find

$$\begin{aligned}
&\text{Norm}\left\{(b-c)\sqrt{a^2+\theta+\alpha} + (c-a)\sqrt{b^2+\theta+\beta} + (a-b)\sqrt{c^2+\theta+\gamma}\right\} \\
&= (b-c)^4\alpha^2 + (c-a)^4\beta^2 + (a-b)^4\gamma^2 - 2(c-a)^2(a-b)^2\beta\gamma - 2(a-b)^2(b-c)^2\gamma\alpha - 2(b-c)^2(c-a)^2\alpha\beta \\
&\quad + 4\theta(a-b)(b-c)(c-a)\left[(b-c)\alpha + (c-a)\beta + (a-b)\gamma\right] \\
&\quad + 4(a-b)(b-c)(c-a)\left[(bc(b-c)\alpha + ca(c-a)\beta + ab(a-b)\gamma\right] \\
&\quad - 4\theta(a-b)^2(b-c)^2(c-a)^2 .
\end{aligned}$$

Hence, first writing $a-x,\ b-x,\ c-x$ in place of a, b, c; then y^2 for θ, and $(-a''^2, -b''^2, -c''^2)$ for (α, β, γ); and finally introducing z for homogeneity, we find

$$\begin{aligned}
&\text{Norm}\left\{(b-c)\sqrt{(x-az)^2+y^2-a''^2z^2} + (c-a)\sqrt{\,,,} + (a-b)\sqrt{\,,,}\right\} = z^2 \text{ into} \\
&\qquad z^2((b-c)^4a''^4 + (c-a)^4b''^4 + (a-b)^4c''^4 \\
&\qquad\qquad - 2(c-a)^2(a-b)^2b''^2c''^2 - 2(a-b)^2(b-c)^2c''^2a''^2 - 2(b-c)^2(c-a)^2a''^2b''^2) \\
&\quad -4y^2(b-c)(c-a)(a-b)\left[\ (b-c)a''^2 + (c-a)b'' + (a-b)c''^2\right] \\
&\quad -4\ (b-c)(c-a)(a-b)\ \{\ (b-c)a''^2\ (z^2bc - zx(b+c) + x^2) \\
&\qquad\qquad\qquad\qquad\qquad + (c-a)b''^2\ (z^2ca - zx(c+a) + x^2) \\
&\qquad\qquad\qquad\qquad\qquad + (a-b)c''^2\ (z^2ab - zx(a+b) + x^2)\} \\
&\quad -4y\ (b-c)^2(c-a)(a-b)^2 .
\end{aligned}$$

so that the equation $(b-c)\sqrt{\mathsf{A}^\circ} + (c-a)\sqrt{\mathsf{B}^\circ} + (a-b)\sqrt{\mathsf{C}^\circ} = 0$, in its rationalised form, contains $(z^2 = 0)$ the line infinity twice, and the curve is thus a conic. If $a''^2 = b''^2 = c''^2 = k''^2$, then the expression of the norm is

$$= z^2 \text{ into } -4(a-b)^2(b-c)^2(c-a)^2(y-k''^2z^2),$$

viz., when the three circles have each of them the same radius k'', the curve is the pair of parallel lines $y^2 - k''^2z^2 = 0$; and in particular when $k'' = 0$, or the circles reduce themselves each to a point, then the curve is $y^2 = 0$, the axis twice.

ANNEX IV.—*On the Trizomal Curves $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$, which have a Cusp, or two Nodes.*

The trizomal curve $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$, has not in general any nodes or cusps: in the particular case where the zomal curves are circles, we have however seen how the ratios $l : m : n$ may be determined so that the curve shall acquire a node, two nodes, or a cusp; viz., regarding a, b, c as current areal co-ordinates, we have here a conic $\frac{l}{\text{a}} + \frac{m}{\text{b}} + \frac{n}{\text{c}} = 0$, the locus of the centres of the variable circle, and the solution depends on establishing a relation between this conic and the orthotomic circle or Jacobian of the three given circles. I have in my paper "Investigations in connection with CASEY's Equation," Quart. Math. Jour. vol. viii. (1867), pp. 334–342, given, after Professor CREMONA, a solution of the general question to find the number of the curves $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$, which have a cusp, or which have two nodes, and I will here reproduce the leading points of the investigation. I remark, that although one of the loci involved in it is the same as that occurring in the case of the three circles (viz., we have in each case the Jacobian of the given curves), the other two loci Σ and Δ, which present themselves, seem to have no relation to the conic of centres which is made use of in the particular case.

We have the curves $U = 0$, $V = 0$, $W = 0$, each of the same order r; and considering a point the co-ordinates whereof are (l, m, n), we regard as corresponding to this point the curve $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$, say for shortness, the curve Ω, being as above a curve of the order $2r$, having r^2 contacts with each of the given curves $U = 0$, $V = 0$, $W = 0$. As long as the point (l, m, n) is arbitrary, the curve Ω has not any node, and in order that this curve may have a node, it is necessary that the point (l, m, n) shall lie on a certain curve Δ; this being so, the node will, it is easy to see, lie on the curve J, the Jacobian of the three given curves; and the curves J and Δ will correspond to each other point to point, viz., taking for (l, m, n) any point whatever on the curve Δ, the curve Ω will have a node at some one point of J; and conversely, in order that the curve Ω may be a curve having a node at a given point of J, the point (l, m, n) must be at some one point of the curve Δ. The curve Δ has, however, nodes and cusps; each node of Δ corresponds to two points of J, viz., for (l, m, n) at a node of Δ, the curve Ω is a binodal curve having a node at each of the corresponding points of J; each cusp of Δ corresponds to two coincident points of J, viz. for (l, m, n) at a cusp of Δ, the curve Ω has a node at the corresponding point of J. The number of the binodal curves Ω is thus equal to the number of the nodes of Δ, and the number of the cuspidal curves Ω is equal to the number of the cusps of Δ; and the question is to find the Plückerian numbers of the curve Δ. This Professor CREMONA accomplished in a very ingenious manner, by bringing the curve Δ into connexion with another curve Σ (viz., Σ is the locus of the nodes of those curves

$lU + mV + nW = 0$ which have a node), and the result arrived at is that for the curve Δ

Order = $3(r-1)(3r-2)$,
Class = $6(r-1)^2$,
Nodes = $\frac{3}{2}(r-1)(27r^3-63r^2+22r+16)$,
Cusps = $3(r-1)(7r-8)$,
Double tangents = $\frac{3}{2}(r-1)(12r^3-36r^2+19r+16)$,
Inflexions = $12(r-1)(r-2)$;

so that, finally, the number of the cuspidal curves $\sqrt{lU} + \sqrt{mV} + \sqrt{nW} = 0$, is $= 3(r-1)(7r-8)$, and the number of the binodal curves of the same form is $= \frac{3}{2}(r-1)(27r^3-63r^2+22r+16)$. When the given curves are conics, or for $r = 2$, these numbers are $= 18$ and 36 respectively; but the formulæ are not applicable to the case where the conics have a point or points of intersection in common; nor, consequently, to the case of the three circles.

II.—*On the Motion, Equilibrium, and Forms of Liquid Films.* By the late Sir David Brewster, K.H., D.C.L., &c. (Plates I. and II.) Communicated by Francis Deas, Esq., LL.B.

(Read 6th April 1868.)

[This paper was transmitted to the Council by Sir David Brewster, on the 8th February 1868, with the following remarks:—" I have tried in vain to finish the two most important of my papers on Liquid Films, but the most beautiful drawings of all the phenomena which its purpose was to describe have been finished. I think, therefore, that my friend Mr Deas will, by means of these drawings, produce an interesting paper. The drawings are numerous and large, but many of them may be reduced in size. As this is the last of my papers, I hope the Council will not grudge the expense of having them well lithographed."

In another letter, Sir David Brewster expresses a wish, that in the event of the paper being printed in the "Transactions," notice should be taken of the fact that the drawings were executed by his friend Miss Dickenson.]

I. *On some Transformations in Films when brought in contact with Surfaces of Glass.**

(1.) Let a film be formed on the rim of a cylindrical wine-glass, at or very near its margin; cover it immediately with a watch-glass, and holding the latter firmly in its place, invert the whole, so that the film is placed in a vertical position. The film will now attach itself to the watch-glass at the lowest point where it is in contact with the margin of the wine-glass, and will run up the concave surface of the former. At the same time, the film will leave the margin of the wine-glass at its upper edge, and retreat into the glass, running down its inner surface. A film of the form of the segment of a sphere will thus be produced, which, with the upper portions of the inner surfaces of the watch-glass and wine-glass, will form a hollow filled with air, as shown in fig. 1. This state of matters will remain the same, in whatever position we now place the wine-glass, the figure which has thus been produced being one of equilibrium. The phenomenon produced arises from the fact, that when the original film is first taken up on the margin of the wine-glass, a drop of liquid always remains in the bottom of the glass, and when the glass is inverted, so as to bring the film into the vertical position, this drop runs down the inner surface of the glass till, reaching its lowest point, it brings the film into close contact with the watch-glass; the film now attaches itself to and spreads itself over the surface of the latter in a similar way to what takes place when a bubble blown from a pipe is brought into contact with any smooth surface, whilst the original system of equilibrium being now disturbed, the upper part of the film is put in motion, and a new system of equilibrium is formed.

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* The experiments under Head 1, are best performed by using a watch-glass of considerable concavity, but they will all succeed more or less perfectly by using a piece of perfectly flat glass, or even by employing the convex instead of the concave surface of the watch-glass, provided we take care that the surface of the film on the wine-glass does not project in any part above its rim.

In making this experiment, the rise of the film upon the surface of the watch-glass is generally so rapid, that it is difficult to watch its progress, but by careful inspection it may be observed to commence first at a single point; *the edge of attachment* of the film to the watch-glass then becomes first elliptical, then circularly concave, then a straight line, then convex, as seen in fig. 2, where the lines represent the edge of the film on the watch-glass in its different stages of progress. The changes which take place in the curvature of the concave spherical film within the wine-glass during these stages are also very curious. The concave film may sometimes be produced without inverting the glass, by holding the watch-glass firmly in position with the thumb, and briskly shaking the glass. The drop of liquid is thus thrown upon the margin of the film.

(2.) The concave spherical film being thus formed, we can reduce matters to their original state (*i.e.* reproduce the single original film on the margin of the wine-glass) by slowly and carefully removing the watch-glass. The experiment is best made by lifting the watch-glass from its connection with the wine-glass at its upper margin, keeping the two in close contact at their lower margin. The upper edge of the concave film will thus again rise in the wine-glass, while its lower edge will descend along the watch-glass; and when it has reached the point where the two glasses are in contact, the watch-glass may be removed, and the original film left on the wine-glass. The edge of the film, as it descends the watch-glass, exhibits the same series of curves as it did in ascending in the last experiment; but, of course, in a reversed order. The nature of these curves may be much more satisfactorily observed in this experiment than in the last, as their progress is much less rapid. Before entirely removing the watch-glass, we can cause the film to ascend or descend at pleasure, with any degree of rapidity, by approximating or separating the two glasses.

In making the first experiment, the drop of fluid which causes the film to attach itself to the wine-glass generally entirely escapes; consequently, if we repeat this experiment upon the film as re-formed in the second experiment, it will seldom succeed, there being no superfluous fluid to produce the necessary contact. If, however, we place a drop of fluid, either on the surface of the reformed film or on the inner surface of the watch-glass, the experiment will succeed, and may be repeated indefinitely, till the film bursts—a fact which clearly proves the drop of fluid to be the agent in producing the phenomenon.

These experiments succeed best with freshly-formed films, but they may be performed even after the colouring matter has formed on the film, though with less certainty. In re-forming the original film, as above described, the wine-glass may be held either horizontally, vertically, or even inverted entirely with the watch-glass held beneath it.

(3.) If, after forming the concave film, as in the first experiment, we raise the watch-glass from the wine-glass, maintaining the contact between the two only at one point, and then cause the watch-glass to rotate by this point round the wine-glass, the concave film will likewise move round the wine-glass; and by continuing the movement, the concave film may be gradually enlarged, till it passes into the original single film, adherent to the wine-glass only as in the last experiment, the curvature of its margin passing through the forms already observed.

(4.) If, having formed the concave film as in the first experiment, we lift the watch-glass perpendicularly from the margin of the wine-glass, the edges of the concave film will remain adherent to the wine-glass and watch-glass respectively, the film being stretched out, while the fluid between the glasses at their previous line of contact will be drawn into a second film, which unites with the concave film to form a cylindrical bag attached above to the watch-glass, and below to the wine-glass.* By continuing to raise the watch-glass, this bag will sometimes become detached from the watch-glass, and return into the form of the concave film; at other times it will leave the wine-glass entirely, and take the form of a lens upon the watch-glass. Fig. 3 exhibits the cylindrical bag thus formed, still attached to both wine-glass and watch-glass.

As seen from the figure, the remainder of the wine-glass is now covered by a separate film, upon which the cylindrical bag partly rests. This may be regarded as that part of the original film which, in the first experiment, ran up and attached itself to the concave surface of the watch-glass, and which, in the process of perpendicularly lifting the watch-glass, has become restored to its original position, and which, for the sake of distinctness, we may call the complementary film. Whether the cylindrical bag will adhere to the watch-glass or return into the concave film, seems to depend on the relation of the radius of curvature of the concave film to that of the film covering the remainder of the glass, and to that of the watch-glass itself. The smaller the concave film, and the more convex the complementary film, the more readily will the bag leave the latter, and attach itself to the watch-glass in the form of a lens, while the less concave the watch-glass, the more difficult it is to produce this result, till, when we use a perfectly flat glass, it will rarely take place in any case. When, however, the complementary film is either accidentally or purposely broken, so that only the concave film remains, the latter will invariably pass into the form of the lens. The curious bag shape of the film in this latter form of the experiment, while thus passing from the wine-glass to the watch-glass, is shown in fig. 4.

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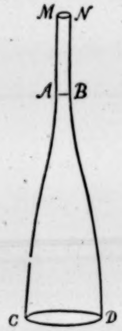
* The experiment succeeds best, by first raising the portion of the watch-glass most distant from the concave film, and then lifting the whole watch-glass vertically.

(5.) The lens, when obtained upon the watch-glass, may be restored to the wine-glass in the form of the concave film, by merely replacing the watch-glass so that the lens shall touch the edge of the wine-glass, and it may be taken up and replaced several times, and that whether the complementary film has remained on the wine-glass or not.

(6.) If, when the complementary film has remained on the wine-glass after removing the lens upon the watch-glass, we replace the lens so as to form the concave film, we can, by lifting the watch-glass, or causing it to rotate, as in the previous experiments, reproduce the original single film, which in this case consists of the same film which was taken up as a lens upon the watch-glass.

II. *On the Motions and Figures of Equilibriums of Films within Single Hollow Cones.*

In the following experiments, a glass cone of the form in the annexed diagram was used—the perpendicular height of the cone being 3 inches, the diameter of the base being 1 inch. The aperture at the apex can be opened or closed at pleasure, by inserting a plug at MN. When this vessel is dipped in the soap solution (MN being open), it gives, when raised, a plane film at CD. This film (as shown in the previous paper on "Liquid Films," vol. xxiv. p. 503, of the "Transactions") does not remain at CD, but at once commences to ascend towards AB, with a velocity increasing—*firstly*, with the angle of the cone itself; and, *secondly*, with the angle of inclination at which the cone is held. Arrived at AB, where the cone ends in the cylindrical tube, the film becomes stationary.



The motions and disposition of films within the cone may be well studied, by blowing small bubbles from a quill or small tube, and inserting them within the vessel. Thus, if a bubble of moderate size be placed on the side of the cone near its base, it will gradually ascend till its convex surface comes in contact with the opposite side of the cone, when it will separate into two plane films, the lower of which will remain stationary at AB, the upper a little way up the tube. This movement of the bubble is represented in fig. 5.

Again, having obtained a plane film, a little way up the cone, insert a small bubble against the side of the cone, touching the under surface of the plane film, the upper surface of the bubble will unite with the adjacent portion of the plane film to form a single film, and produce the system shown in fig. 6. Insert now, on the opposite side, a second bubble of the same size with the first, and we obtain the system shown in fig. 7, the upper surface of each of the two hollow figures thus produced being convex, the lower concave, their surface of contact being a plane film. The same result is produced, though no plane film is first formed within the cone, by simply inserting two bubbles of equal size, so as to come in contact with one another. In a similar manner, three bubbles may be inserted in place of two, the result is the system shown in fig. 8, where the three bubbles are united to one another by three plane films meeting in a central vertical straight line. The system produced by inserting four bubbles in a similar manner to the last, is a very curious and beautiful one, peculiarly interesting on account of its analogy to the system already described in the previous paper, as formed within the wire cube (*vide* "Transactions," vol. xxiv. p. 505). This system, shown in fig. 9, consists of four similar figures, the curvatures of whose sides are extremely curious, united in their centre, just as in the case of the system within the wire cube, by a plane quadrilateral film bounded by convex lines. In the case of the wire cube, it was seen that this plane film could be transposed from the vertical to the horizontal position, by blowing upon its margin through a small tube. It is difficult, if possible, to repeat the experiment in this form upon the similar system within the cone, but the same result may be produced, *i.e.* the system may be obtained with the plane film, in the horizontul position, by means of the following process.

It has been seen that in the case of the systems formed of two and three bubbles respectively, it is immaterial whether we proceed by first obtaining a plane film upon which the bubbles are inserted, or by simply inserting the bubbles upon the sides of the cone without the previous existence of the plane film. But in the case of the system now under consideration, if we proceed by first inserting a plane film as a basis upon which to place the four bubbles, and take care that these are not too large, the result is the system shown in fig. 10, in which it will be seen the plane quadrilateral film connecting the four bubbles, instead of being vertical, is horizontal, consisting in fact of the central portion of the original plane film which has remained *in situ*, the remainder of it having amalgamated with the films of the bubbles. A beautiful variety of this experiment may be made in the following manner:—Having obtained the system with the plane film in the horizontal position, withdraw the plug from MN, so as to allow the system to rise in the cone (a result which can always be insured by gently sucking out the air by the mouth from MN); the system being thus contracted, the sides of the bubbles are brought in contact with one another, the horizontal film is first contracted to a point, and then (the system being now unstable) passes into the vertical position. By again blowing air in at MN, the system may be made again to descend, and the plane film to pass once more into the horizontal position.

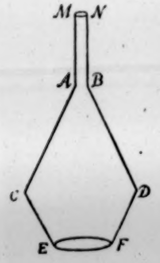
It will be remembered that, in the case of the wire cube, a beautiful system was produced by forming a small cubical film in the centre of the polyhedron (*vide* "Transactions," vol. xxiv. Plate xxxiv. Fig. 4). This, it was seen, could be produced either by dipping the cube a second time in the solution, or by blowing a small bubble in the centre of the polyhedron. A precisely similar result may be obtained with the similar system within the cone, and that by either of the same methods. The result of causing this system to ascend the cone is strikingly beautiful; the cube becomes narrowed above, assuming the form of a truncated pyramid; and if caused still further to ascend, the horizontal film at its apex entirely disappears, and a perfect pyramid is produced, resembling that shown in Plate xxxiv. Fig. 10, of vol. xxiv. of "The Transactions."*

At the same instant that this takes place, the four original bubbles become united above the pyramid in a vertical straight line, but this system being unstable, the line quickly passes into a vertical plane film.

As the system continues still further to ascend the cone, it generally settles into one or other of the forms already described, *i.e.*, two, three, or four similar hollow figures united by one or more plane vertical films. At any time before the pyramid disappears, we can again reduce the system to its original form by blowing air in at *MN*. This experiment is best made with a cone whose vertical angle is more obtuse than that described.

III. *On the Motions and Figures of Equilibrium of Films within Double Hollow Cones.*

In these experiments a glass vessel was used of the form of that in the annexed diagram, consisting of two cones united by their bases. The vessel, as in the last case, can be closed or opened by a plug fitting at *MN*. When *MN* is open, and the vessel is dipped into the soap solution, it gives, when raised, a single *plane* film at *EF*. When the vessel is dipped so that the liquid rises in it a little below *CD*, and *MN* is then entirely or nearly closed, we obtain, on raising the vessel so that the liquid runs out, a convex film below *CD*, and a concave one at *EF*.



If we make the same experiment with the vessel dipped so deep that the liquid rises above *CD*, we obtain a variety of regular or irregular systems of films, which, when destroyed, often leave a fine concave film above *CD*. On opening *MN*, this concave film rises to *AB*, the rings or bands of colour rising to higher orders by the thickening of the film. This concave film is sometimes formed along with and above the regular or irregular systems of films, and separate from them.

When a film is formed exactly at *CD*, it is fixed, and moves neither towards *AB* nor *EF*; but if it be made slightly to advance towards *AB*, it will rise to *AB*, or if made to advance in the direction of *EF*, it will fall to *EF*, and there become stationary. Regular binary, ternary, or quaternary systems, similar to those already described as formed within the single cone (the quaternary system having the plane film in its centre, sometimes vertical, sometimes horizontal), may be produced by simply dipping the vessel in the solution and raising it with *MN* closed; but these systems can all be obtained with much greater ease and certainty by blowing small bubbles within the vessel in the same way as was done with the single cone. Figs. 11 to 14 exhibit these systems respectively. The system formed by thus inserting two bubbles of equal size is shown in fig. 11; that by inserting three such, in fig. 12; that by inserting four, in fig. 13. This last system is, like that formed within the single cone, united in its centre by a plane quadrilateral film, which, as in the experiment with the single cone, may be obtained in the horizontal instead of in the vertical position, by introducing a plane film at or near *CD*, before inserting the bubbles, *vide* fig. 14. All the preceding experiments with single and double cones may be performed with cones of any angle, the forms of the curvatures of the figures produced being modified accordingly.

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* The so-called cube is of course more or less a truncated pyramid from the first, owing to the conical form of the vessel used.

These experiments may likewise be further varified by using cylindrical tubes instead of cones. If a system of three bubbles be adopted, a small bubble inserted in their centre will take the form of a triangular prism;* with four, a cube; with five, a pentagonal prism; with six, a parallelopiped, and so on; or two cubes, two parallelopipeds, &c., may be inserted one above the other, forming systems analogous to that in Plate xxxiv. Fig. 5, of vol. xxiv. of "The Transactions." If two cubes be thus inserted into the system formed in the single cone, and the system be then allowed to rise in the cone, the lower cube will retain more or less its quadrilateral form, while the upper passes into the pyramid—a state of things which may be compared to that previously described with the wire pyramid, and shown in Plate xxxiv. Fig. 11, of vol. xxiv. of "The Transactions."

When a cylinder is used, the vertical lines of the prism, cube, &c. thus introduced are straight lines, their horizontal lines being outwardly convex.

IV. *On some Miscellaneous Experiments on Paraboidal, Conical, and Cylindrical Films.*

If the cone previously described be dipped in the solution, so that it is filled to *AB*, and *MN* be then closed, we obtain, on raising the cone, a paraboidal film at *CD*, and a plane film about half way up the cone (fig. 15). The paraboloid, as the excess of fluid escapes from its apex, becomes first a hemisphere, then the segment of a sphere, and then bursts.

If two very convex films be placed in contact with one another, they will become united by a more or less flattened film, which, as already shown in the previous paper on this subject (Vol. xxiv. p. 503, of the "Transactions"), will be plane, convex, or concave, according to the relative convexity of the bubbles themselves. If we now draw the bubbles gently apart, they will form two truncated cones united by a small circular film at their apices, which may sometimes be reduced to a mere point before bursting, thus forming two perfect cones, as shown in fig. 16.

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* This figure is best obtained by inserting the small bubble before inserting the *third* of the large bubbles.

If, instead of bringing the two convex films into direct connection with one another, we unite them by means of a small bubble placed between them, this intermediate small bubble will assume the barrel shape shown in figs. 17 and 18. If we now decrease the distance between the large convex films, the small bubble will assume the flattened form of fig. 17. If we increase the distance between the large films by drawing them apart, the small bubble will become elongated, as in fig. 18. The larger end of the barrel will always be that in connection with the film of least convexity; in other words, with the film whose radius of curvature is the greater, and when the system is drawn apart so as to break the chain of connection, it is to this film (that, namely, of greater convexity) that the small bubble will always adhere.

The form of the barrel is always that of a more or less perfect cylinder, all of whose bounding lines are convexly curved.

When the large convex films have their radii exactly equal, the barrel-shaped cylinder will be perfectly symmetrical, *i.e.*, the circles at its upper and lower ends will be equal. When this is the case, or nearly so, the system can be drawn apart without suffering disruption till the ends of the cylindrical bubble become almost mere points.

To make the two large films of *exactly* equal convexity is of course impossible practically, but theoretically it would appear that in such a case the small central bubble ought to part connection with each of the large bubbles simultaneously, and fall, resuming its original form of a perfect sphere.

If a cylindrical or conical tube, such as that used in the previous experiments, be slightly dipped into the solution, and then gradually raised vertically, the film formed on its lower aperture will remain attached to the surface of the liquid. The curvatures of the film as we gradually raise the tube are very curious. At first, while the surface of attachment is large, the film is convexly curved; its sides then become straight lines; then concave, then, just before parting with the surface of the liquid, they become slightly convex above, concave below. In fig. 19 these changes of curvature are shown.

If the liquid into which we dip the tube be shallow and small in quantity (that, for example, contained in a watch-glass), it will be elevated bodily when we raise the tube, its whole surface becoming more and more convex as the film becomes more concave. If we insert a film half way up the tube before making this experiment, it will sink in the tube whilst the lower film is convex, and again rise when the lower film begins to become concave.

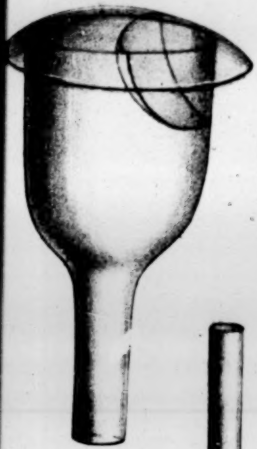


FIG. 1.

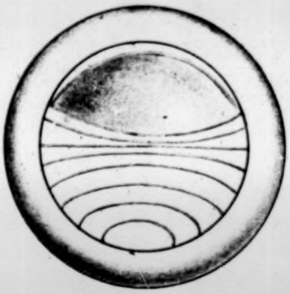


FIG. 2.

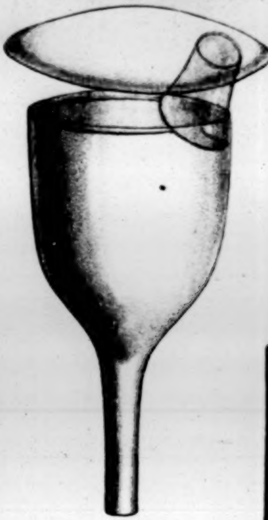


FIG. 3.

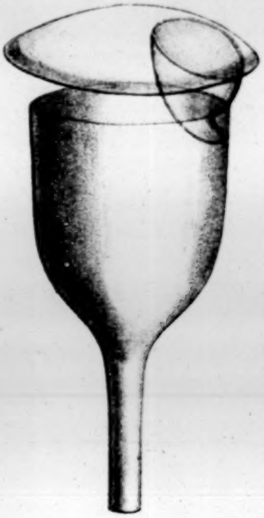


FIG. 4.

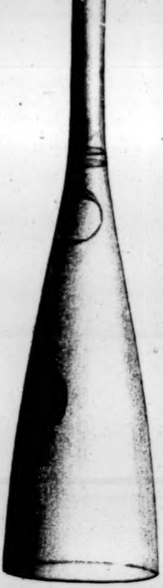


FIG. 5

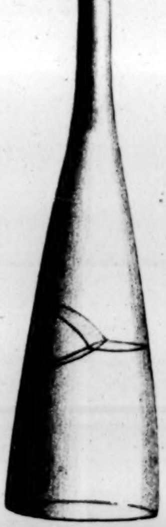


FIG. 6.

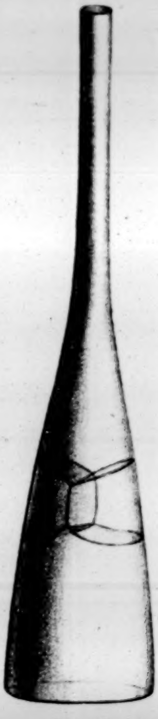
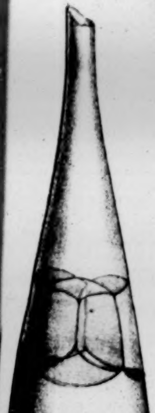
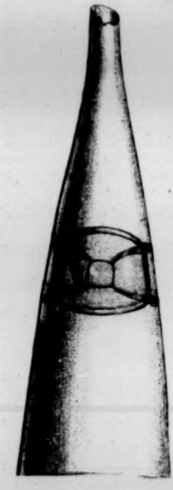
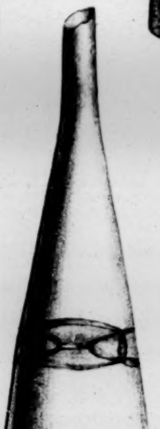
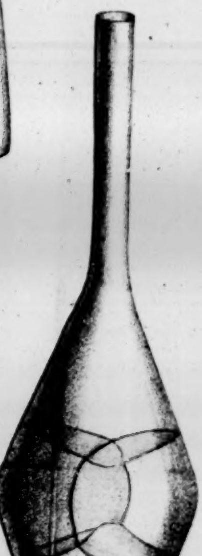


FIG. 7.









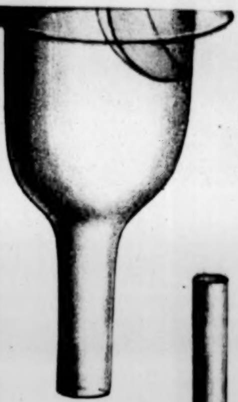


FIG. 1.

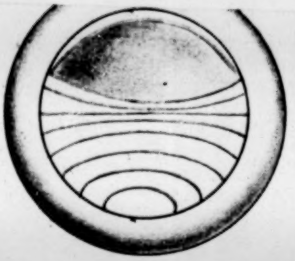


FIG. 2.

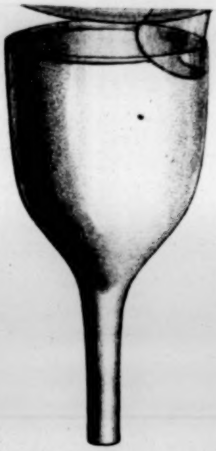


FIG. 3.

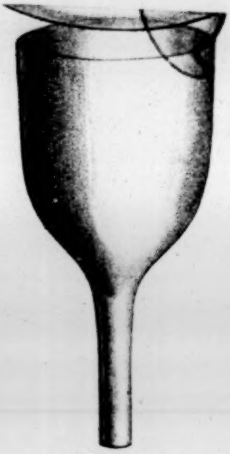


FIG. 4.

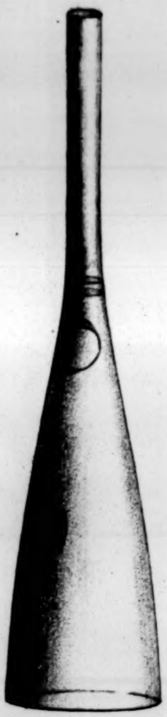


FIG. 5

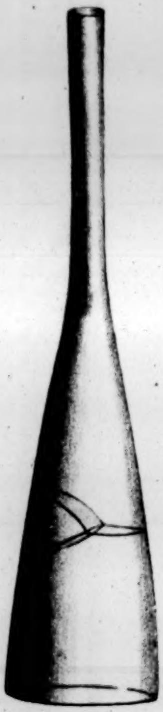


FIG. 6.

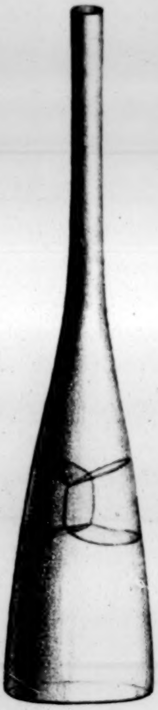


FIG. 7.

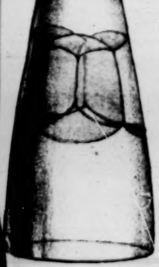


FIG. 8.

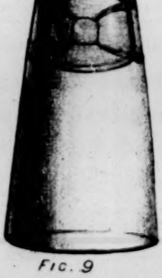


FIG. 9

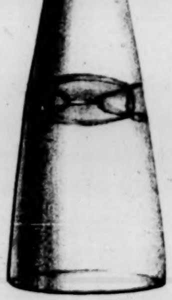


FIG. 10

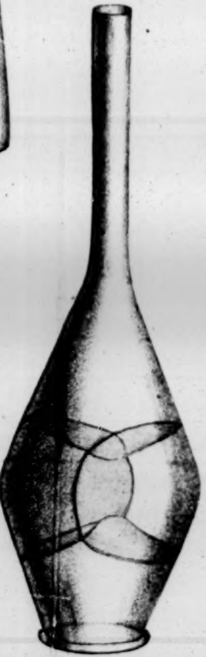


FIG. 11

& Tod, Lith.; Edinr

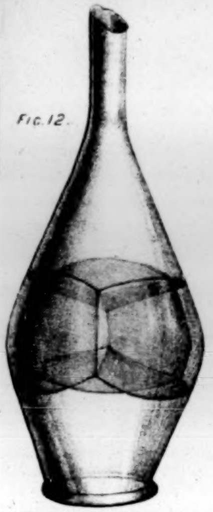


FIG. 12.

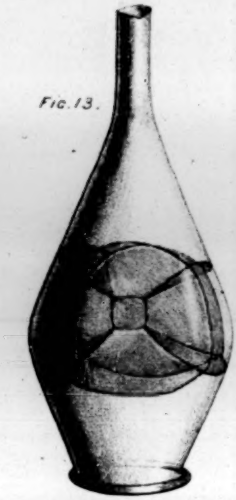


FIG. 13.

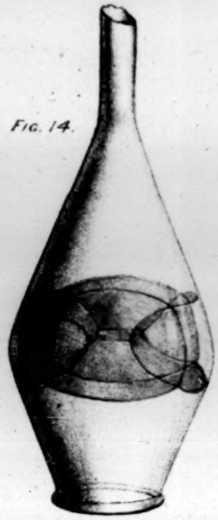


FIG. 14.

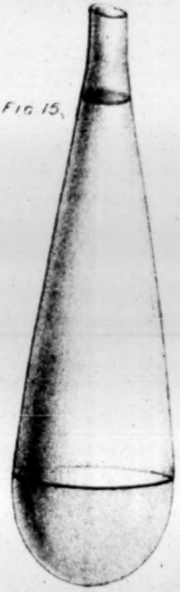


FIG. 15.

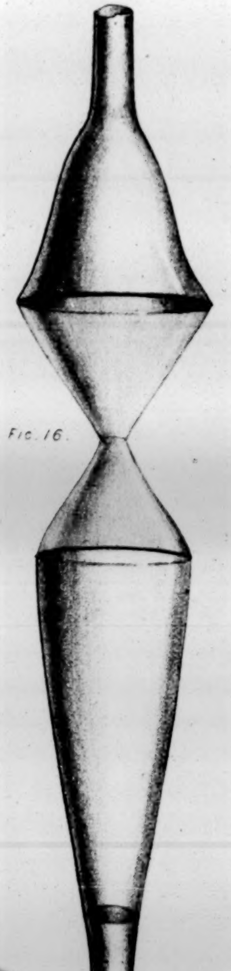


FIG. 16.

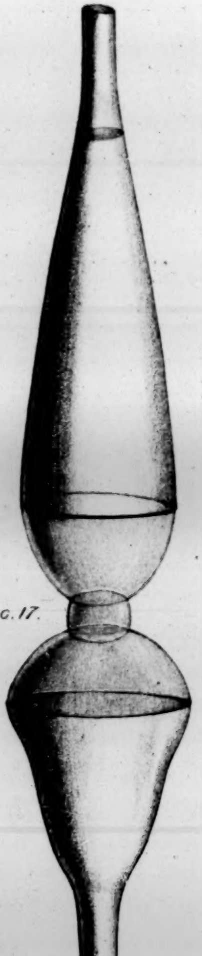


FIG. 17.

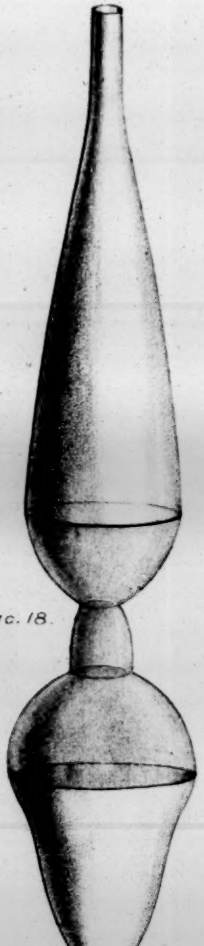


FIG. 18.

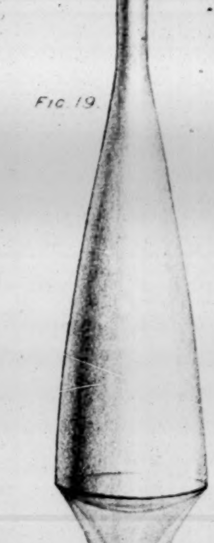


FIG. 19.

FIG. 12.

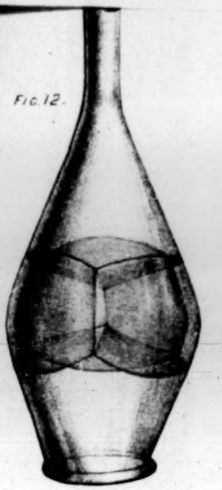


FIG. 13.

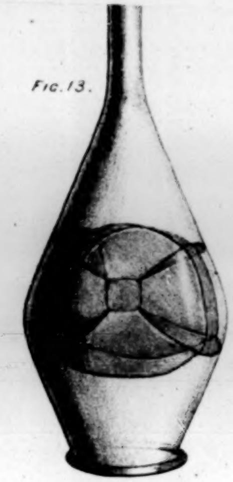


FIG. 14.

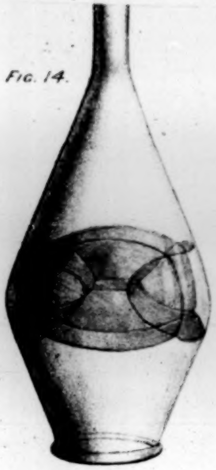


FIG. 15.

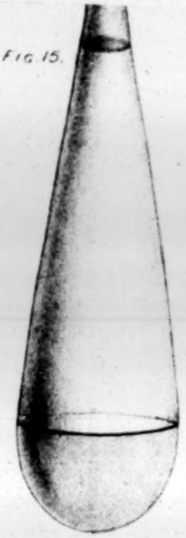


FIG. 16.

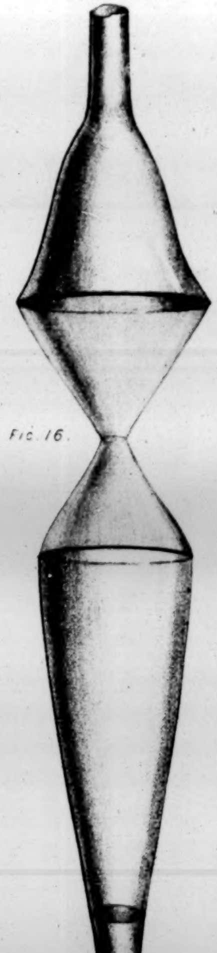


FIG. 17.

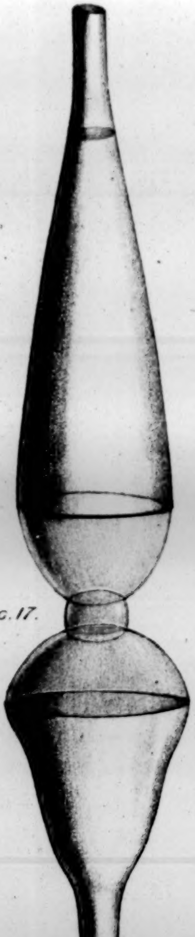


FIG. 18.

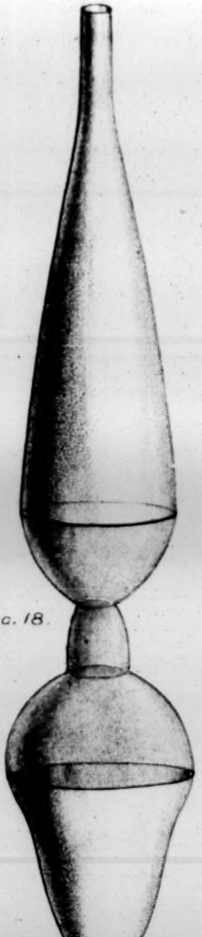
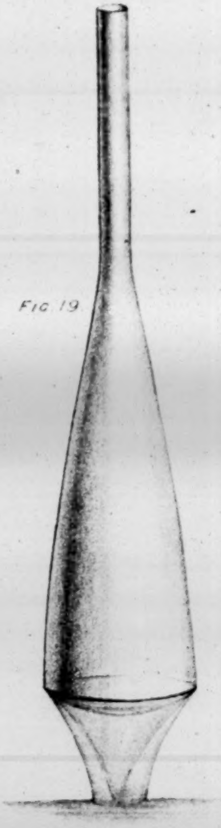


FIG. 19.



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III.—*On the Temperature of the Common Fowl* (Gallus domesticus). By

JOHN DAVY, M.D., F.R.SS. Lond. and Edin.

Read 17th February 1868.

During the last three years I have made a large number of observations on the temperature of the common fowl under different circumstances, the results of which I now beg leave to submit to the Society, with the hope that they may be considered not altogether uninteresting to the physiologist.

The fowls tried were chiefly of the pure Dorking breed. At the time they appeared to be healthy, and all in good condition. They had all the run of a field adjoining the poultry yard.

In all the trials the same thermometer was used,—each degree of which, that of Fahrenheit, was divided into ten parts, and had been warranted correct by the makers after comparison with a standard. The quantity of mercury in the bulb of the instrument was so small that in a minute or two, when introduced into the rectum of the bird, it reached the maximum; and in every instance the rectum was the part of which the temperature was ascertained, presuming it to be there the same as that of the interior of the body generally.

Though the temperature of the air was mostly noted down, as well as other circumstances likely to affect the results, I do not think it necessary to enter into minute details respecting them, partly for the sake of brevity, and partly from their not appearing to influence materially the results. I may, however, remark that the observations were mostly made, whatever the season of the year, between 10 and 11 A.M., and that in the majority of instances the birds had been kept in confinement during the night and early morning, and had not been fed since the day preceding.

1. *Of the Average Temperature.*

The total number of observations made during the whole period on birds of different sexes and different ages, varying in age from five weeks to five years, the majority from six months to two years, amounted to 163; the average temperature deducible from them was 107°·81.

2. *Of the Temperature of the Male and Female.*

The number of males tried was 68, of females 95. The average temperature from the former was 108°·39; from the latter, 107°·36. The highest temperature observed in any one instance of the males was 110°—this in August, in sultry weather, when the thermometer in the shade was 81°; the lowest was 106°·5, whilst the highest noticed in the other sex was 109°·25, the lowest 105°—this in a hen on the sixteenth day of her incubation.

3. *Of the Temperature of the Sexes previous to Maturity.*

The term maturity is used as implying the stage at which the female begins to lay and the male to exercise the generative faculty—in the instance of the former about the sixth or seventh month, in that of the latter a month or two earlier.

The number of both sexes tried was 31,—of males, 16; of females, 15,—yielding conjointly an average temperature of 108°·5, and separately, in the instance of the males, 108°·4; in that of the females, 108°·66. The highest male temperature was 109°·25, the lowest 107°·5; whilst the highest female temperature was 109°·5, the lowest 108°.

4. *Of the Temperature of the Mature Male.*

The temperature of the same male was taken during one year monthly, with the omission of one month. At the beginning the bird was two years old; at the end, when three years old, it weighed nine pounds and a half. During the whole time it seemed in vigorous health. The results were the following:—

In October,	109°·5	In May,	108°·5
November,	109°·5	June,	109°·5
December,	108°·5	July,	108°·75
January,	108°·25	August,	110°·00*
February,	107°·75	September,	108°·77†
April,	108°·25		

affording an average temperature of 108°·77.

5. *Of the Temperature of the Female whilst Laying.*

The number of females tried was 12, varying in age from six or seven months to four or five years. The average temperature reducible from them was 107·5; the lowest temperature noticed in any one was 105°·5—this in a hen five years old; the highest, 108°·5.

6. *Of the Temperature during Incubation.*

Of 14 sitting hens tried, the average temperature was found to be 107°; the lowest temperature observed was 105°; the highest, 109°·5.

The average weight of the fowls at the beginning of sitting was 5 lbs. 13 oz.; the average loss at the end of the process was 1 lb. 7 oz.; the smallest loss in any one instance was 4 oz.—this in that of the fowl the temperature of which at the end was 109°·5; the greatest, and the same in two instances, was 1 lb. 8 oz., and of both the temperature was 107°.

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* Thermometer in shade, 81°.

† Moulting.

7. *Of the Temperature during Moulting.*

Of 10 hens undergoing this change the average temperature was 108°·44 ; the highest in any one instance was 109°·5 ; the lowest, 107°. It is noteworthy that the highest degree was observed in the middle stage of the process, when the surface of the abdomen was nearly destitute of feathers.

Conclusions.

The results, I would beg to remark, are offered merely as approximations; as such they seem to show—

1. That the temperature of the common fowl is 107°·81.

2. That the temperature of the sexes before maturity is comparatively high, being 108°·5, whilst that of the two sexes at this stage varies very little.

3. That the temperature of the male, on the whole, irrespective of any particular age, is higher than that of the female, being as 108°·39 to 107°·3.

4. That the temperature of the fully matured male (a single instance) is 108°·77.

5. That the temperature of the laying hen is 107°·4.

6. That during incubation the temperature falls, and is as low as 107°.

7. And that during moulting it rises to 108°·44.

The variations of temperature, from the highest to the lowest, noticed in the several instances, are less perhaps than might have been expected. No doubt, they were connected with peculiarities of circumstance, which very careful observation might possibly have detected,—circumstances of weather, not only as to temperature, but also as to moisture of air, degree of exposure, and varying strength of wind, not to insist on other conditions, such as the quantity of food taken, the period of fast, amount of exercise, the precise state of health. The very few observations I have made, tending at all to illustrate modifying circumstances in relation to temperature, are the following:—

A hen, seven months old, after having been confined in a basket thirty-six hours without food, the thermometer part of the time below the freezing point, was of the temperature 106°·5.

Another of the same age, confined without food twenty-six hours, the thermometer during the time between 40° and 50°, was 107°.

An old hen, the leg of which had been broken two or three days previous to trial, and had ate little since the preceding evening, was 105°.

A fowl labouring under difficulty of breathing of some hours, was 100°.

A male bird, in full vigour, of the temperature 108°·75, was let loose, and driven till it stopped, after four minutes,—now its temperature was found to be 109°.

On the principal results, as given in the summary, I shall comment but little. Some of them, especially the near equality of the temperature of the sexes before maturity, and the lowering of the temperature of the female at the period of laying, may be viewed as physiological problems; others, as the loss of weight during incubation, with a fall of temperature, and the contrary as to temperature during the period of moulting, seem in harmony with what might be expected according to the theory of animal heat. In the one case, there being a loss of substance of the individual in connection with diminished nutriment, without apparently a febrile disturbance of health; whilst in the other case, it would seem there was such a disturbance, with an elevation of temperature, such as is witnessed in febrile diseases.

The change of temperature that takes place in the chick on quitting the egg is remarkable, and, as it appears to me, strongly in support of that view of animal heat, in which respiration and the formation of carbonic acid by the union of oxygen with carbon, is considered its principal source. In the egg, just before hatching, the chick is of a temperature rarely exceeding 100°, and that derived more from the incubating mother than from the organic changes in progress; but no sooner is the hatching completed, and the young bird freely respires, than there is a sudden elevation of temperature. In one instance, in which I had an opportunity of watching a gosling in the hatching act, the temperature actually rose, and that suddenly, from about 100° to 106°, after the manner of a hybernating animal, such as the dormouse, in which in passing from its torpid to a state of activity without taking food, there has been, as I have noticed, a rise of temperature from 56° to 99°·5.*

It is also remarkable how soon the young fowl, after becoming tolerably fledged and capable of securing adequate food, which would seem to be simultaneous with the consumption of the internal provisional yolk, its earliest nourishment, it attains a comparatively high temperature. Thus of 11—7 of them thirty-one days old, 4 thirty-five—the average temperature in the middle of October was 108°·73; the lowest in any one instance 108°, the highest 109·5: and I have found the temperature of nestlings also comparatively high,—a young swallow, fully fledged, just after being taken from the nest on the 28th July, was 108°.

Dr William Edwards, in his very interesting work, "On the Influence of the Physical Agents on Life," has come to the conclusion that "the power of producing heat in warm-blooded animals is at its maximum at birth, and increases successively until adult age,"—a conclusion which seems to me questionable, and requiring, if admitted at all, to be received with many restrictions. Some of the results I have obtained are opposed to it, and others might be mentioned tending to invalidate it.

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* *See* Physiological Researches (1863), p. 85.

IV.—*On the Burning Mirrors of Archimedes, with some Propositions relating to the concentration of Light produced by Reflectors of different forms.* By John Scott, Esq., Tain. (Plate III.)

(Read 6th January 1868).

As the reputed fact of Archimedes having burned the Roman ships engaged in the siege of Syracuse, by concentrating on them the solar rays, has not only been doubted but disbelieved by some of the most eminent scientific men, I shall briefly give the evidence on both sides.

The burning of the ships of Marcellus is mentioned by most of the ancient writers who refer to the machines which Archimedes employed in the defence of his native city, and their statements have been repeated by succeeding authors, without any doubts having been expressed until comparatively recent times. Our earliest authorities on the subject are Diodorus Siculus, Lucian, Galen, Dion, and Pappus. At a later period the architect Anthemius, of Tralles, in a fragment entitled περι παραδοξων μηχανηματων (Wonderful Machines), not only dwells particularly on the burning mirror of Archimedes, but adds, besides, that it was universally admitted in his time that Archimedes had destroyed the Roman fleet by means of burning mirrors. It is also mentioned by Hero, a writer on military engines—about the middle of the seventh century—and by Eustathius, Zonares, and Tzetzes, who flourished in the twelfth. The last two have transmitted to us passages extracted from the work of Pappus on the siege of Syracuse, which was then extant, but has since disappeared. We give that by Tzetzes as the more definite and circumstantial:—"When Marcellus had placed the ships a bow-shot off, the old man (Archimedes) contrived a hexagonal mirror. He placed at proper distances from the mirror smaller mirrors of the same kind, and which were moved by means of their hinges and certain square plates of metal. He placed it in the midst of the solar rays at noon both in summer and winter. The rays of the sun being reflected by this, a dreadful fire was excited on the ships, which reduced them to ashes at the distance of a bow-shot."

In the sixteenth century mirrors similar to that of Archimedes seem to have engaged the attention of Leonhard Digges, and of Baron Napier of Merchiston. At a subsequent period Father Kersher took up the same subject, prosecuting it with such assiduity that he travelled to Sicily to examine the coast in the vicinity of Syracuse, and came to the conclusion that Archimedes might have opportunities of placing his mirror within 30 yards of the ships. He also mentions in his "Magica Catoptrica," as the result of experiments which he had performed, that the superimposed rays reflected from five plane mirrors at the distance of more than 100 feet, produced a heat which could scarcely be endured. Apparently convinced of the practicability of the achievement by means of plane mirrors, he entreats future mathematicians to prosecute the subject. BUFFON, following in his steps, completely established the fact that combustible materials can be set on fire at distances corresponding to the accounts we have of the mirror of ARCHIMEDES. This he effected by means of a combination of plane reflectors, consisting of ordinary looking-glasses, 8 inches by 6, attached to a single frame, each glass, as well as the supporting frame, being capable of motion in every direction. With forty of these glasses he set on fire tarred beech at a distance of 66 feet. A plank, smeared with tar and brimstone, was ignited at 126 feet by 98 glasses. A combination of 128, with a clear sun, inflamed very suddenly a plank of tarred fir at 150 feet, the conflagration springing up at once over a space of 16 inches in diameter—the whole reflected image of the sun at that distance. In addition to these experiments made about the beginning of April, others were exhibited with the summer sun, by which wood was kindled at 200 and 210 feet, and silver and other metals were melted at distances varying from 25 to 40 feet.

Let us now consider the evidence on the opposite side of the question. DESCARTES and others have treated the whole affair as fabulous, from the belief that the burning glass must have consisted of a single spherical or parabolic reflector. But since no mention is made of the kind of specula employed by ARCHIMEDES, such objections, after the successful experiments of BUFFON, necessarily become irrelevant. Another and quite different ground of doubt has arisen from the circumstance, that POLYBIUS, LIVY, and PLUTARCH make no mention of the destruction of the Roman fleet by means of burning glasses, although they describe somewhat in detail the ballistæ and other military engines constructed by ARCHIMEDES to resist the assailants. How far the silence of the forementioned writers should be taken as negative evidence, it is not easy to determine. It seems to indicate, at least, that the damage effected by the burning mirror was confined to comparatively narrow limits; for, if prominent in the defence, we naturally expect that it would have received special notice. But allowing that the fleet sustained no serious injury from this source, some fact must be left sufficient to account for the belief prevalent among ancient authors in favour of the achievement. Nothing less, I conceive, will suffice than to admit that ARCHIMEDES made an attempt to destroy the Roman fleet in the manner described. Such an admission, however, implies that he must have previously tried his combination of reflectors in private, and was able to ignite combustible substances at considerable distances. The mere attempt to bring an artillery so singular and subtle to bear on the fleet is in itself a conclusive proof that an experiment, similar to those exhibited by BUFFON in the Jardin-du-Roi at Paris, had been successfully performed 2000 years before within the walls of Syracuse. We can no more suppose the contrary than believe that any of the nations of modern Europe would send into the field of actual warfare a novel piece of ordnance without subjecting it to a previous trial. Though on this hypothesis the element of success as an engine of war is questionable, the invention of the mirror as a fact in the history of science remains entire. Neither should it be forgotten, as perhaps a reason for the silence of POLYBIUS, LIVY, and PLUTARCH, that with the fall of Syracuse and the death of the illustrous inventor, all definite information relating to the scientific principles of the mirror seems to have perished—a result not improbable, when we consider that its application to the art of war would induce the original possessors to retain its construction, as far as possible, a secret in their own hands. Finally, it is an admitted axiom in estimating historical evidence, that the silence of one author respecting an event is never considered sufficient to invalidate a plain and consistent statement of that event made by another. That our conclusions should be formed in strict accordance with the principle enunciated, may be made apparent by striking and well-ascertained facts, some of which have been inaccurately recorded, and others altogether omitted by the most reliable contemporary historians. As an instance of the former, modern authorities maintain that the account given by LIVY of the route by which HANNIBAL conducted his army across the Alps cannot be reconciled with that by POLYBIUS, and an extract from Sir CHARLES LYELL's "Principles of Geology" will show an historical omission equally inexplicable. Speaking of the first eruption of Vesuvius, he says, "The younger PLINY, although giving a substantial detail of so many physical facts, and describing the eruption and earthquake and the shower of ashes which fell at Stabiæ, makes no allusion to the sudden overwhelming of two large and populous cities, Herculaneum and Pompeii. In explanation of this omission, it has been suggested that his chief object was simply to give TACITUS a full account of the particulars of his uncle's death. It is worthy of remark, however, that had the buried cities never been discovered, the accounts transmitted to us of their tragical end might well have been discredited by the majority, so vague and general are the narratives, or so long subsequent to the event. TACITUS, the friend and contemporary of PLINY, when adverting in general terms to the convulsions, says merely cities were consumed or buried. . . . . SUETONIUS, although he alludes to the eruption incidentally, is silent as to the cities. They are mentioned by MARTIAL in an epigram as buried in cinders; but the first historian who alludes to them by name is DION CASSIUS, who flourished about a century and a half after PLINY."

Returning to BUFFON's combination of reflectors—when the focus had to be changed, a numerous staff of assistants required about half an hour to re-adjust the mirrors. After all, the superposition of the reflected light would be imperfect, each operator being liable to mistake the deviation of the image reflected by some other glass for that of which he had charge. PEYRARD has appended to his translation of the works of ARCHIMEDES a memoir of his own, in which he calls attention to these defects, and to remedy them proposes that a telescope and somewhat complex apparatus be attached to each reflector, that with fewer hands the adjustment might be rendered more accurate and speedy.

It is obvious that the improvement suggested by PEYRARD is only partial and its success doubtful. I conceive the Archimedean mirror to have been a compound reflecting apparatus, free from these defects, capable of being directed by one eye and guided by a single hand. From TZETZES we learn that the mirror was hexagonal, that like BUFFON's it consisted of a combination of reflectors, and that at proper distances from the outer mirror were placed other smaller ones of the same kind. This last peculiarity of the Archimedean mirror has no parallel in that of BUFFON; and yet the arrangement of the smaller mirrors, at proper distances from the larger, indicates that the relation of the two kinds to each other formed an essential feature of the combination. Although the above passage conveys no information respecting the nature of the specula, there is such a description of the connection of the parts as an intelligent observer might carry away, and yet be unacquainted with the scientific principles involved in the construction.

In the sequel (Arts 13 and 14) we show how larger and smaller specula, all of the same kind, can be so connected as to form a single compound reflector capable of concentrating on a single spot the reflected rays, and of darting them instantaneously in any direction, when they will produce the effects ascribed to the mirror of ARCHIMEDES. The results thus being the same, and the construction of the combination coinciding with the description given by TZETZES, we therefore infer that the real principle of the Archimedean mirror has been attained, and that the accounts which have come down to us respecting it are in the main authentic. This will be brought out more fully after the following general propositions have been considered.

For the historical facts contained in the preceding, I am chiefly indebted to PEYRARD's edition of the "Works of Archimedes," and to the article "Burning Glasses," in the Encyclopædia Britannica.

ARTICLE I.—Prop. *When the Light emanating from a Luminous Sphere of small angular diameter falls on a very small Plane Mirror; to find the Intensity of the reflected Light at any distance from the Mirror.*

Let GH (fig. 1, Plate III.) be the mirror, B the point of reflection, AC the small luminous sphere, and DE the plane on which the reflected light falls. It follows, as a consequence of the equality of the angles of incidence and reflection, that the angle which DE subtends at B is equal to that which AC subtends, DE being supposed perpendicular to the cone of rays reflected from B.

Whatever may be the form of the infinitesimal mirror at B, the perpendicular section of the reflected rays, at all finite distances from B, is a circle.

Let α = the angle ABC = DBE.

r = radius of the sphere AC.

c = the distance from B, measured along the slant side of the cone of rays, at which the diameter of the circle of reflected light is equal to unity.

I = the intensity of the light radiating from the sphere at its surface.

I' = intensity of the light at the mirror, on a plane perpendicular to the axis of the cone of rays falling on B.

$\therefore k\mathrm{I}'$ = intensity of the reflected light at the mirror, k being a constant less than unity.

d = distance of the centre of the sphere from B.

d' = distance of DE from B, measured along the slant side of the cone of reflected rays.

$\delta\mathrm{A}$ = sectional area of the light incident on small mirror at B.

Area of the circle DE $= \pi d'^2 \sin^2\frac{\alpha}{2}$ . . . . . (1),

" $= \frac{\pi d'^2}{4c^2}$ . . . . . (2),

because $c : d' :: \frac{1}{2}$: radius of the circle DE.

Since the intensity of the light, emanating from the surface of a luminous sphere and falling on a concentric spherical surface, is inversely as the square of the distance from its centre,

$$r^2 : d^2 :: \mathrm{I}' : \mathrm{I} \quad \therefore \mathrm{I} = \frac{d^2\mathrm{I}'}{r^2} \quad . \quad . \quad . \quad (3).$$

But $$\sin\frac{\alpha}{2} = \frac{r}{d} = \frac{1}{2c} \quad \therefore \mathrm{I} = \frac{\mathrm{I}'}{\sin^2\frac{\alpha}{2}} \quad . \quad . \quad . \quad (4)$$

$$= 4c^2\mathrm{I}' \quad . \quad . \quad . \quad (5).$$

Intensity of the reflected light at DE $= \frac{k\mathrm{I}'\delta\mathrm{A}}{\text{area of circle DE}}$ . . (6),

" $= \frac{k\mathrm{I}'\delta\mathrm{A}}{\pi d'^2 \sin^2\frac{\alpha}{2}}$ . . . (7).

" $= \frac{4c^2k\mathrm{I}'\delta\mathrm{A}}{\pi d'^2}$ . . . (8).

When the plane on which the reflected light falls is not perpendicular to the axis of the cone of rays, as de (fig. 1), the intensity of the light at e will exceed the intensity at d from the greater obliquity of the rays at the latter point. But if the angle DBE be small, and the plane de cut the axis of the cone at a considerable angle, the intensity will be nearly uniform over the whole ellipse de; and the centre of the ellipse may be viewed as situated in the axis of the cone, because the elongated cone approaches nearly to a cylinder.

$$\therefore \text{intensity on } de = \frac{k\mathrm{I}'\partial\mathrm{A}}{\text{area of ellipse DE}} \quad . \quad . \quad (9).$$

In the case of the sun's light the above formulæ will give pretty accurate approximations, since $a = 32'$ and $c = 107{\cdot}4$ nearly . . . (10).

Article 2.—Prop. *When a Cylindrical Beam of Solar Light is reflected from a Plane Mirror; to find the Intensity on a Plane Surface perpendicular to the direction of the reflected Beam, and at any distance from the Mirror.*

Since the rays which emanate from any single point in the sun's disc may be considered as perfectly parallel, however large the mirror, it follows that those from the centre of the disc will, after reflection from the mirror, form a perfectly cylindrical beam of parallel rays, and will cast on the given plane a circle of light of uniform intensity.

Let ABG (fig. 2) represent this circle, each point in its area is the centre of a circular image of the sun's disc reflected from a corresponding point, or infinitesimal area, of the mirror.

If O be one of these points, and the circle ADBP the image of the sun's disc at the given distance, or, in other words, the base of a cone of rays whose apex is at the mirror, as shown in fig. 1, Art. 1, the illumination or intensity of the light at O will be that due to the superposition of all the images of the sun's disc whose centres fall within the area ACBPA; for none of the images of the sun's disc, whose centres fall without the above area, can extend so far as O. The point O is therefore illuminated by a portion of the incident beam equal in area to ACBPA, every increment of which, after reflection, gives rise to a conical pencil of rays, a part of whose base overspreads the point O.

Let I' = intensity of the solar light on a plane perpendicular to the direction of the incident beam.

Let $\mathrm{AO} = r$, $\mathrm{FA} = \rho$, $\mathrm{FO} = z$, $\mathrm{AFO} = \theta$, $\mathrm{AOC} = \phi$, and $u_1\ u_2\ u_3\ u_4\ u_5$, &c., the sectional areas, at the mirror, of the respective pencils whose light overspreads the point O. Whatever may be the form of these small increments, the base of the cone of light to which they give rise will be a circle at all finite distances from the mirror, as shown Art. 1.

Therefore the intensity of the solar light at O from area $u_1 = \frac{k\mathrm{I}'u_1}{\text{circle ADBP}} = \frac{k\mathrm{I}'u_1}{\pi r^2}$,

$$\text{,,} \qquad u_2 \qquad = \frac{k\mathrm{I}'u_2}{\pi r^2},$$

$$\text{,,} \qquad u_3 \qquad = \frac{k\mathrm{I}'u_3}{\pi r^2},$$

$$\text{,,} \qquad \text{,,} \qquad \text{,,}$$

$$\text{,,} \qquad u_n \qquad = \frac{k\mathrm{I}'u_n}{\pi r^2}.$$

Whole intensity at O $= \frac{k\mathrm{I}'}{\pi r^2}(u_1 + u_2 + u_3 \ldots\ldots + u_n)$;

$$= \frac{k\mathrm{I}' \times \text{area ACBPA}}{\pi r^2} \qquad . \quad . \quad . \quad . \qquad (1),$$

$$= \frac{k\mathrm{I}'}{\pi r^2}(2 \times \text{sector AFC} - 2 \text{ triangle AOF} + 2 \times \text{sector AOP}),$$

$$= \frac{k\mathrm{I}'}{\pi r^2}\left\{ \rho^2\theta + r^2(\pi - \varphi) - \rho z \sin\theta \right\} \qquad . \quad . \quad . \qquad (2),$$

$$= \frac{k\mathrm{I}'}{\pi r^2}\left\{ \rho^2\theta + r^2\left(\pi - \sin^{-1}\left(\frac{\rho\sin\theta}{r}\right)\right) - \rho z \sin\theta \right\},$$

because $\sin\varphi = \frac{\rho\sin\theta}{r}$,

also $z = \rho\cos\theta - r\cos\varphi$,

$$= \rho\cos\theta - \sqrt{(r^2 - \rho^2\sin^2\theta)}.$$

When $\rho > r$ in the preceding expressions, and $\theta = 0$, then $\phi = 0$, $z = \rho - r$, and intensity at O $= \frac{k\mathrm{I}'}{\pi r^2} \times \pi r^2 = k\mathrm{I}'$ . . . . (5).

If, therefore, the circle ADPB falls wholly within the circle ABG, the intensity of the illumination on the circular space, whose centre is F and radius $\rho - r$, is constant and equal to that of the beam when it leaves the mirror, or the intensity is the same as if all the rays from the sun's disc were parallel to one another.

When $\rho = r$, then, $z = 0$, and F is the only point whose intensity $= k\mathrm{I}'$.

Next when $\rho < r$ and $\theta = \pi$, then $\phi = \pi$, $z = r - \rho$, and intensity $= \frac{k\mathrm{I}'}{\pi r^2} \times \rho\pi^2$ $= \frac{k\mathrm{I}'\rho^2}{r^2}$ . . . . (6).

This formula expresses the intensity throughout the circle, whose radius is $r - \rho$.

When $z = r + \rho$, the intensity vanishes.

The preceding limiting cases are also evident from the geometrical consideration of the problem taken in connection with equation 2; and the same principles can be applied to find the intensity, whatever the shape of the perpendicular section of the incident beam.

ARTICLE 3.—When there are n mirrors similar to the preceding, the light from each making an angle of incidence i, with the perpendicular to the plane on which it is thrown, the intensity of the central spot in each of the preceding cases becomes—

$$\text{In 1st case, } nk\mathrm{I}' \cos i, \quad . \quad . \quad . \quad . \quad (1).$$

$$\text{In 2d case, } \frac{nk\mathrm{I}'\varrho^2 \cos i}{r^2}, \quad . \quad . \quad . \quad . \quad (2).$$

ARTICLE 4.—Prop. *A Small Luminous Sphere has its centre in one of the foci of a Prolate Elliptic Mirror, to find the Intensity on any Small Plane surface situated in the other Focus.*

Let f (fig. 3) represent the luminous sphere.

β = the angle which the small plane at F makes with FX: the axis of z coinciding with the axis of the mirror, and the plane of xz being perpendicular to the small plane passing through F; the axis of y, which is at right angles to the plane of xz, will therefore coincide with the small plane which passes through F.

r = FP.

θ = the angle PFZ.

ϕ = the angle which the projection of r on the plane of xy make with FX.

a = radius of the luminous sphere at f.

V = the angle which r makes with the normal to the small plane at F.

I = the intensity of the light at the surface of the small sphere.

Since the well-known differential of a volume $r^2 \sin\theta\, d\theta\, d\phi\, dr$ has for its perpendicular section, at the surface of the spheroid, $r^2 \sin\theta\, d\theta\, d\phi$; we may suppose the whole surface of the elliptic mirror to be divided into small areas, each of which receives from the sphere at f, and reflects to F, a pencil of light, whose perpendicular section, at any point P of the mirror, is $r^2 \sin\theta\, d\theta\, d\phi$.

Moreover, the intensity at P perpendicular to $\mathrm{P}f = \frac{a^2\mathrm{I}}{\mathrm{P}f^2}$ (by Art. 1, Equa. 3), and after reflection it becomes $\frac{a^2k\mathrm{I}}{\mathrm{P}f^2}$.

The cone of rays reflected from the increment of surface at P, in the direction PF, will have expanded at F into a circle (perpendicular to the radius vector), whose radius by similar triangles is $\frac{ar}{\mathrm{P}f}$ and area $\frac{\pi a^2 r^2}{\mathrm{P}f^2}$.

By Art. 1. Equa. 6, the intensity of the light on this circle at $\mathrm{F} = r^2 \sin\theta\, d\theta\, d\phi \times \frac{a^2k\mathrm{I}}{\mathrm{P}f^2} \div \frac{\pi a^2 r^2}{\mathrm{P}f^2} = \frac{k\mathrm{I}}{\pi} \sin\theta\, d\theta\, d\phi$.

and $\frac{k\mathrm{I}}{\pi}\sin\theta\, d\theta\, d\phi \cos \mathrm{V}$ = its intensity at F on the small plane, whose equation is $z\cos\beta - x\sin\beta = 0$, ... (1),

$\cos \mathrm{V} = \frac{z}{r}\cos\beta - \frac{x}{r}\sin\beta = \cos\beta\cos\theta - \sin\beta\sin\theta\cos\phi$.

Substituting for cos V its value we obtain for the whole intensity at F, on the small plane,

$$\frac{k\mathrm{I}}{\pi}\iint(\cos\beta\cos\theta - \sin\beta\sin\theta\cos\phi)\sin\theta\, d\theta\, d\phi, \quad . \quad . \quad (2).$$

Denoting this integral by u,

$$u = \frac{k\mathrm{I}}{2\pi}\int(\cos\beta\sin^2\theta + \sin\beta\sin\theta\cos\theta\cos\phi - \theta\sin\beta\cos\phi)\, d\phi + \mathrm{C}.$$

Now $\cot\theta = \tan\beta\cos\phi$, when $\mathrm{V} = 90°$, that is, when PF coincides with the plane denoted by Equation 1,

$$\therefore \theta = \cot^{-1}(\tan\beta\cos\phi)$$

$$\sin\theta = \frac{1}{(1+\tan^2\beta\cos^2\phi)^{\frac{1}{2}}}$$

$$\cos\theta = \frac{\tan\beta\cos\phi}{(1+\tan^2\beta\cos^2\phi)^{\frac{1}{2}}}$$

Taking the integral between the limits $\theta = 0$, and $\theta = \cot^{-1}(\tan\beta\cos\phi)$,

$$u = \frac{k\mathrm{I}}{2\pi}\int\left\{\frac{\cos\beta(1+\tan^2\beta\cos^2\phi)}{1+\tan^2\beta\cos^2\phi} - \cot^{-1}(\tan\beta\cos\phi)\sin\beta\cos\phi\right\}d\phi.$$

$$= \frac{k\mathrm{I}}{2\pi}\left\{\int\cos\beta\, d\phi - \sin\beta\sin\phi\cot^{-1}(\tan\beta\cos\phi) + \int\frac{\sin\beta\tan\beta\sin^2\phi\, d\phi}{1+\tan^2\beta\cos^2\phi}\right\}$$

$$= \frac{k\mathrm{I}}{2\pi}\left\{\int\cos\beta\, d\phi - \sin\beta\sin\phi\cot^{-1}(\tan\beta\cos\phi) - \int\cos\beta\, d\phi + \int\frac{\sec\beta . d(\tan\phi)}{\sec^2\beta + \tan^2\phi}\right\}$$

$$= \frac{k\mathrm{I}}{2\pi}\left\{\tan^{-1}(\cos\beta\tan\phi) - \sin\beta\sin\phi\cot^{-1}(\tan\beta\cos\phi)\right\} + \mathrm{C}.$$

and between the limits $\phi = 0$ and $\phi = 90°$,

$$u = \frac{k\mathrm{I}}{2\pi}\left(\frac{\pi}{2} - \frac{\pi}{2}\sin\beta\right),$$

$= \frac{k\mathrm{I}}{4}(1 - \sin\beta)$, which is the intensity at F of the light reflected from that portion of the spheroid bounded by the planes passing through the co-ordinate axes of $+x$ and $+z$, $+y$ and $+z$, and the small plane produced.

Similarly $\frac{k\mathrm{I}}{4}(1 + \sin\beta)$ gives the intensity of the light, from that portion enclosed by the planes passing through the axes $-x$ and $+z$, $+y$ and $+z$, and the small plane at F produced. Hence the intensity of all the light which can fall on the side of the small plane towards A (fig. 3).

$$= 2\left\{\frac{k\mathrm{I}}{4}(1 - \sin\beta) + \frac{k\mathrm{I}}{4}(1 + \sin\beta)\right\},$$

$$= k\mathrm{I}, \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Also $\frac{kI}{2}\sin\beta$ is the intensity at F due to the light reflected from the part of the mirror intercepted between the small plane produced, and the co-ordinate plane of xy.

In the same manner it can be shown, that the intensity of the light concentrated at F, on the opposite side of the small surface, and reflected from the remaining portion of the spheroid, is also equal to kI.

These results are independent of a, the radius of the luminous sphere, and are equally true for all spheroids which have F for one of their foci, wherever the other may be situated.

It appears then, in conclusion, that the light emanating from a small luminous sphere, with its centre in one of the foci of a prolate elliptic mirror, produces at the other focus a nucleus of radiant light and heat, equal in intensity to the radiation at the luminous surface diminished by the quantity lost by reflection.

Again, putting $\beta = 0$ (in Equation 2), we obtain intensity

$$= \frac{kI}{\pi}\iint \sin\theta\cos\theta\,d\theta\,d\varphi\,, \qquad (4).$$

Integrating between $\varphi = 0$, and 2π;

intensity
$$= 2\pi \times \frac{kI}{\pi}\int \sin\theta\cos\theta\,d\theta\,,$$

$$= 2kI\int \sin\theta\cos\theta\,d\theta\,;$$

and between θ°, and zero,

$$= kI\sin^2\theta\,, \qquad (5).$$

This expresses the intensity at the focus of the light reflected from a segment of the spheroid intercepted between the vertex and a plane perpendicular to the axis; and the intensity produced by a zone intercepted between two planes, perpendicular to the axis of revolution, is

$$kI\,(\sin^2\theta - \sin^2\theta') \qquad (6).$$

Article 5.—The preceding proposition is true, independently of the size and form of the luminous body in the focus f (figs. 3 and 4).

Since radiant light and heat diminish in the inverse ratio of the square of the distance, it follows that the quantities received from circular areas of equal angular magnitudes are equal, whatever their absolute magnitudes, when the intensities at the radiating surfaces are equal. Taking this principle in connection with the fact, that a luminous surface appears equally bright when viewed at any angle, the light emanating from CD, part of the surface of DCE, will therefore have at F the same intensity as if it had proceeded from the small sphere AB (fig. 4). But the light reflected from P to F can only emanate from some part of the surface DC, which lies within the cone described by PA, revolving about Pf. Hence the intensity at F of the light radiating from DC, and reflected at P, is equal to that which would have resulted from the sphere AB; and the same is true for every increment of the elliptic mirror. Therefore, the total illumination at F, from the luminous surface DCE, is equivalent to that from the small sphere AB.

From this and the preceding article, we infer that a zone or segment of an elliptic reflector may be used as a pyrometer. For if such a zone, contained between two planes perpendicular to the axis, be placed before an opening in a furnace, the place of the focus *f* falling within the heated body, the heat reflected to F may be reduced, by diminishing the breadth of the zone, until it can be measured by a Fahrenheit thermometer; and I, the intensity of the total radiation from any point *f* within the furnace, can be determined in degrees of Fahrenheit by Art. 4, Equa. 6.

ARTICLE 6.—Prop. *When a Parabolic Reflector has its axis directed to the centre of the Sun, to find the intensity of the converging Rays which fall on a small Plane Disc at the Focus.*

Let a = angular diameter of the sun, which is about 32′,

c = the distance at which the reflected image of the sun expands into a circle equal to unity in diameter, being about 107·4,

r = PF (fig. 5),

I′ = intensity of the sun's rays at the earth's surface,

I = intensity at the surface of the sun,

$u, k, \beta, \theta, \phi$, and V = the same as in proposition (Art. 4).

Then $\frac{\pi r^2}{4c^2}$ = area of the circle, which the light reflected at P occupies at F perpendicular to PF.

The intensity of the light reflected from the increment of surface at P on this circle by Art. 1, Equa. 6.

$$= r^2 \sin\theta \, d\theta \, d\varphi \times k\mathrm{I}' \div \frac{\pi r^2}{4c^2},$$

$$= \frac{4c^2 k\mathrm{I}'}{\pi} \sin\theta \, d\theta \, d\varphi ;$$

and on the small disc at F, $= \frac{4c^2 k\mathrm{I}'}{\pi} \sin\theta \, d\theta \, d\varphi \cos \mathrm{V}$.

$$\cos \mathrm{V} = \cos\beta \cos\theta - \sin\beta \sin\theta \cos\varphi.$$

$$\therefore u = \frac{4c^2 k\mathrm{I}'}{\pi} \iint (\cos\beta \cos\theta - \sin\beta \sin\theta \cos\varphi) \sin\theta \, d\theta \, d\varphi, \qquad (1).$$

Integrating as in Article 4, we obtain for the intensity of the light reflected from the corresponding sections of the parobolic mirror

$$c^2 k\mathrm{I}' (1 - \sin\beta , \quad . \quad . \quad . \quad . \qquad (2).$$

and $$c^2 k\mathrm{I}' (1 + \sin\beta), \quad . \quad . \quad . \quad . \qquad (3).$$

Hence the total intensity at F (fig. 5) on the side of the plane disc towards

A, resulting from the light reflected from the segment of the paraboloid cut off by the plane of the disc produced,

$$= 2\{c^2k\mathrm{I}'(1 - \sin\beta) + c^2k\mathrm{I}'(1 + \sin\beta)\} = 4c^2k\mathrm{I}' = k\mathrm{I} \quad . \qquad (4),$$

I being the intensity at surface of the sun.

But $c = 107$ nearly, therefore the numerical value of this equation

$$= 45796k\mathrm{I}'$$

nearly, which is a degree of concentration several times that of the most powerful burning glass ever constructed.

Again putting $\beta = 0$ (in Equation 1),

$$u = \frac{4c^2k\mathrm{I}'}{\pi}\iint \sin\theta\cos\theta\, d\theta\, d\phi\,;$$

and integrating as in Article 4, Equation 4,

$$u = 4c^2k\mathrm{I}'\sin^2\theta\,; \quad . \quad . \quad . \quad . \qquad (5),$$

which gives the intensity at the focus of the light reflected from a segment of a paraboloid, intercepted between the vertex and a plane perpendicular to the axis; and the intensity produced by a zone, intercepted between two planes perpendicular to the axis, is

$$4c^2k\mathrm{I}'(\sin^2\theta - \sin^2\theta') = k\mathrm{I}(\sin^2\theta - \sin^2\theta') \quad . \quad . \qquad (6).$$

Equation 5 shows that the concentration at the focus varies as $\sin^2\theta$: it is a maximum when $\theta = 90°$, and is independent of the parameter of the parabola. It may therefore be inferred that a reflector employed to detect the heat of the lunar rays should be as large a segment of a paraboloid as possible; and the same condition is essential in improving to its utmost limit the space-penetrating power of the reflecting telescope.

Again, suppose the parabolic mirror to extend to infinity, it can also be shown that the light concentrated at the focus on the other side of the small disc is equal to $4c^2k\mathrm{I}' = k\mathrm{I}$. What has been proved respecting the intensity at the focus is approximately true for every point on the plane of the small disc not farther from F than $\frac{p}{8c}$, the quantity $\frac{p}{8c}$ being the radius of the sun's image reflected from the vertex of the paraboloid, and p the parameter of the generating parabola. Thus, in every position in space, when the axis of a parabolic mirror, whose extent of surface is not less than that cut off by a plane passing through the focus, is directed to the sun, a circle of radiant light and heat is formed equal in intensity to the radiation at the solar surface minus the quantity lost by reflection.

Article 7.—The intensity at the focus of a parabolic reflector is independent both of the form and distance of the luminous body.

Let BAC (fig. 6) represent a section of the parabolic reflector, and GH that of a luminous surface of uniform intensity; it can be shown, as in the case of the spheroid (Art. 5), that the concentration at F produced by the only rays which can fall on it, namely, those emanating from GH parallel to the axis AF, is equal to the intensity at the luminous surface GH, minus the quantity lost by reflection. It is evident that the section of the luminous body must not be less than CB.

Article 8.—Prop. *When the Axis of a Mirror in the form of a Right Cone is directed to the centre of the Sun, to find the Intensity of the reflected Light on any point in a Plane placed perpendicular to its Axis.*

Let CAD (fig. 7) represent a section of the mirror,

O the point on which we wish to determine the intensity of the reflected light.

Every small increment of the mirror gives rise to a cone of rays which casts an ellipse of light on the plane at F, the major axis of which passes through the point F. The light of all these ellipses, whose centres fall within a certain distance of the point O, will overspread it and increase the intensity at that point.

If P (fig. 8) be the centre of one of these ellipses NOM, considerably magnified, whose circumference passes through O, then P is a point in the curve within which must fall the centres of all the ellipses whose light can overspread O.

To find the equation to this curve,

let $FP = \rho$

$FO = z$

α and β = the co-ordinates of P, referred to rectangular axes whose origin is at O,

θ = the angle PFO.

Now $a^2y^2 + b^2x^2 = a^2b^2$ is the equation to the ellipse NOM, the centre being the origin.

When referred to the axes OX and OY; by substituting,

$$y = (y' - \beta)\cos\theta - (x' - \alpha)\sin\theta\,,$$

$$x = (y' - \beta)\sin\theta + (x' - \alpha)\cos\theta\,,$$

$$\tan\theta = \frac{\beta}{z+\alpha}\,,\text{ we obtain}$$

$$(a^2\cos^2\theta + b^2\sin^2\theta)(y'-\beta)^2 + (a^2\sin^2\theta + b^2\cos^2\theta)(x'-\alpha)^2 - 2(a^2-b^2)(y'-\beta)(x'-\alpha)\sin\theta\cos\theta = a^2b^2.$$

Putting $x'=0$, $y'=0$, there results the equation to the required curve, which is the locus of the centres of all the preceding ellipses whose transverse axes intersect in F and their circumferences in O,

$$(a^2\cos^2\theta + b^2\sin^2\theta)\beta^2 + (a^2\sin^2\theta + b^2\cos^2\theta)\alpha^2 - 2(a^2 - b^2)\alpha\beta\sin\theta\cos\theta = a^2b^2 \qquad (1),$$

$$\therefore (a^2 + b^2\tan^2\theta)\beta^2 + (a^2\tan^2\theta + b^2)\alpha^2 - 2(a^2 - b^2)\alpha\beta\tan\theta = a^2b^2(1 + \tan^2\theta).$$

Substituting for $\tan\theta$ its value, we obtain

$$\{a^2(z+\alpha)^2 + b^2\beta^2\}\beta^2 + \{a^2\beta^2 + b^2(z+\alpha)^2\}\alpha^2 - 2(a^2 - b^2)(z+\alpha)\alpha\beta^2 = a^2b^2\{(z+\alpha)^2 + \beta^2\},$$

$$\therefore a^2z^2\beta^2 + b^2\{\beta^2 + \alpha(z+\alpha)\}^2 = a^2b^2\{(z+\alpha)^2 + \beta^2\} \quad . \quad . \quad . \quad . \qquad (2).$$

By substituting in this $z + \alpha = \rho\cos\theta$, and $\beta = \rho\sin\theta$, we have the polar equation to the curve, F being the origin,

$$a^2z^2\varrho^2\sin^2\theta + b^2(\varrho^2\sin^2\theta + \varrho^2\cos^2\theta - z\varrho\cos\theta)^2 = a^2b^2\varrho^2,$$

$$\therefore a^2z^2\varrho^2\sin^2\theta + b^2\varrho^2(\varrho - z\cos\theta)^2 = a^2b^2\varrho^2,$$

and

$$\varrho = z\cos\theta \pm a\left(1 - \frac{z^2}{b^2}\sin^2\theta\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \qquad (3).$$

It is evident that the form of the curve represented by the Equations 2 and 3 will vary with the relative values of the constants a, b, and z; but in every case it is symmetrical with respect to the axis FX (fig. 8).

When $z = 0$, the point O is situated on the axis of the mirror, and the curve becomes the circle $\rho = a$.

And if $z = b$, the equation breaks up into two circles whose centres lie in the line FO, and which touch one another externally at the point F, their diameters being $a + b$ and $a - b$ respectively.

When the point F falls without the curve, the radius vector becomes a tangent for the value $\sin\theta = \frac{b}{z}$.

Putting $\theta = 0$ in Equation (1), we get for a plane mirror

$$a^2\beta^2 + b^2\alpha^2 = a^2b^2,$$

an ellipse with O for its centre, and equivalent to

$$a^2y^2 + b^2x^2 = a^2b^2.$$

Now, if PHG (figs. 9 and 10) represent the curve, considerably magnified, expressed by equation 3, in deducing the intensity at O the proposition divides itself into cases depending on the relative positions of the points F and O (as in figs. 9 and 10).

Case 1. When the distance FO from the axis (fig. 7) is so small compared with FB that the distance of any point within the curve similar to PHG from CB, that part of the mirror where the light which overspreads O is reflected, may be considered equal to FB.

Let i = angle of incidence on the mirror,
R = perpendicular distance of B from the axis,
r = perpendicular distance of any point in the curve similar to HPG (fig. 9), as represented by FQ,
v = FA = FB, because BFA is an isosceles triangle (fig. 7),
u = intensity at O,
I' = intensity of the sun's light at the earth's surface;
then $2i$ = angle BFA,
$90^\circ - i$ = angle BAF,
$a = b \sec 2i$,
$\pi ab = \pi b^2 \sec 2i$ = area of ellipse, semi-axes a and b,
z = FO, in this case less than b.

Since the rays reflected from C and D (fig. 7) fall upon the same point O, the circumference of the circle described by O in its revolution about the axis is illuminated by the light reflected from the two annuli described by C and D. Besides the point O is situated so near the axis, that the perpendicular distances of C and D from the axis may be considered as equal to one another. Thus, to find the concentration which results from the converging of the rays to the axis, we have,

$$2\pi r : 4\pi R :: I' : \text{intensity at distance } r \text{ from F}, \left(= \frac{2RI'}{r}\right),$$

the intensities being estimated on planes perpendicular to the rays. Wherefore the intensity on the plane at F of the reflected light

$$= k \times \frac{2RI'}{r} \times \cos 2i = \frac{2kRI' \cos 2i}{r},$$

which would give the intensity on the increment at Q (figs. 9 and 10), if the sun's rays were perfectly parallel. But instead of this light being confined to the increment at Q, it is spread over an ellipse whose area $= \pi b^2 \sec 2i$, and hence the intensity at O due to this increment

$$= \frac{2k\,RI' \cos 2i}{r} \times \frac{r\,dr\,d\theta}{\pi b^2 \sec 2i} = \frac{2k\,RI' \cos^2 2i}{\pi b^2}\,dr\,d\theta\,;$$

and the same is true for a corresponding increment on the other side of the axis GH (figs. 9 and 10),

$$\therefore u = \iint 2 \times \frac{2k\,RI' \cos 2i}{r} \times \frac{r\,dr\,d\theta}{\pi b^2 \sec 2i}.$$

$$= \frac{4k\,RI' \cos^2 2i}{\pi b^2} \iint dr\,d\theta \quad . \quad . \quad . \quad (7).$$

This, taken between the limits $r = \rho$, and $r = 0$; and $\theta = \pi$, and $\theta = 0$, gives the total intensity

$$u = \frac{4k\,\mathrm{RI}'\cos^2 2i}{\pi b^2}\int_0^\pi \rho\, d\theta\,.$$

Now, from Equa. 3, $\rho = z\cos\theta + a\left(1 - \frac{z^2}{b^2}\sin\ ^2\theta\right)^{\frac{1}{2}}$, z being less than b,

$$\therefore u = \frac{4k\,\mathrm{RI}'\cos^2 2i}{\pi b^2}\int_0^\pi \left\{a\left(1 - \frac{z^2}{b^2}\sin^2\theta\right)^{\frac{1}{2}} + z\cos\theta\right\}d\theta\,.$$

Expressed in terms of elliptic functions,

$$u = \frac{4k\,\mathrm{RI}'\cos^2 2i}{\pi b^2}\left[a\,\mathrm{E}_{\frac{z}{b}}\,(\theta) + z\sin\theta\right] + \mathrm{C}\,.$$

$$= \frac{4k\,\mathrm{RI}'\cos^2 2i}{\pi b^2}\,a\,\mathrm{E}_{\frac{z}{b}}\,(\pi) \quad . \quad . \quad . \quad . \qquad (8).$$

But by Art. 1, R cosec $2i : c :: b : \frac{1}{2}$, (R cosec $2i$ being = FB, fig. 7),

$$\therefore b = \frac{\mathrm{R}\operatorname{cosec} 2i}{2c}\,,$$

$$= \frac{\mathrm{R}}{2c\sin 2i}\,,$$

and

$$a = b\sec 2i = \frac{b}{\cos 2i}\,.$$

Substituting in Equation 8, we get

$$u = \frac{8ck\,\mathrm{I}'\sin 2i\cos 2i}{\pi}\,\mathrm{E}_{\frac{z}{b}}\,(\pi)\,,$$

$$= \frac{4ck\,\mathrm{I}'\sin 4i}{\pi}\,\mathrm{E}'_{\frac{z}{b}}\,(\pi) \quad . \quad . \quad . \quad . \quad . \qquad (9).$$

An expression which may be put in the following form:—If a circle be described with F as a centre (figs. 7, 8, or 9) and $2b$ as a diameter, and an ellipse with O for one of its foci and the same diameter as a major axis; the circumference of the circle will be to the circumference of the ellipse as the intensity at the axis to the intensity at O.

When $z = 0$, then $\mathrm{E}_{\frac{z}{b}}\,(\pi) = \pi$

and

$$u = \frac{4ck\,\mathrm{I}'\sin 4i}{\pi} \times \pi = 4ck\,\mathrm{I}'\sin 4i\,, \quad . \quad . \qquad (10),$$

which is the expression for the intensity at the axis, and shows that in the same conical mirror it is constant at every point in the axis; whereas, in conical mirrors of different inclinations, there are two which produce a maximum effect at the axis, viz., when $4i = 90^\circ$ and 270° (in Equa. 10), that is, when BAF (fig. 7) $= 67\frac{1}{2}^\circ$ or $22\frac{1}{2}^\circ$.

When

$$z = b \text{ (Equa. 9)}, \ E_{\frac{z}{b}} (\pi) = 2 ,$$

and

$$u = \frac{8ck\mathrm{I}' \sin 4i}{\pi} \quad . \quad . \quad . \quad (11).$$

If $k = \frac{1}{2}$, the maximum intensity at the axis expressed numerically is

$$u = 4ck\mathrm{I}' \sin 90 = 4 \times 107 \times \tfrac{1}{2}\mathrm{I}' = 214\,\mathrm{I}' .$$

When $k = \frac{1}{2}$ and $4i = 2^\circ\, 36'$, it can be shown by Equa. 10 that $u = 10\mathrm{I}'$ nearly, a heat sufficient to ignite wood and other combustibles. This can be effected at a distance of 130 feet with a segment of the reflector 18 inches broad, and having a mean diameter of 6 feet.

Case 2. When the point O is situated at a considerable distance from the axis, z being much greater than b, the distances CO and DO will now differ perceptibly from one another and from $\mathrm{FA} = \mathrm{FB} = v$ (fig. 7).

Let R and R′ represent the distances of the points C and D respectively from the axis, then

$$\mathrm{R} = f\mathrm{C} \sin 2i = (v + z \cot 2i) \sin 2i ,$$

and

$$\mathrm{CO} = f\mathrm{C} - f\mathrm{O} = \mathrm{FA} + \mathrm{F}f - f\mathrm{O} = v + z \cot 2i - z \operatorname{cosec} 2i = v - z \tan i .$$

Similarly for the point D,

$$\mathrm{R}' = (v - z \cot 2i) \sin 2i ,$$

and $\mathrm{DO} = v + z \tan i$.

The ellipses of light which overspread O reflected from each increment of the space around C have for their minor and major axes respectively,

$$b = \frac{v - z \tan i}{2c} , \qquad a = \frac{v - z \tan i}{2c \cos 2i} ;$$

and the corresponding quantities with respect to the space around D are

$$b' = \frac{v + z \tan i}{2c} , \quad a' = \frac{v + z \tan i}{2c \cos 2i} .$$

It can be shown, as in *Case* 1, Equa. 7, that the concentration at O, due to the light reflected from the space around C, is

$$= \frac{2k\,\mathrm{RI}'\cos^2 2i}{\pi b^2}\iint dr\,d\theta\text{, and taken between } \varrho_1 \text{ and } \varrho_2\text{,}$$

$$= \frac{2k\,\mathrm{RI}'\cos^2 2i}{\pi b^2}\int(\varrho_1-\varrho_2)\,d\theta\,,$$

$$= \frac{4k\,\mathrm{RI}'a\cos^2 2i}{\pi b^2}\int\left(1-\frac{z^2}{b^2}\sin^2\theta\right)^{\frac{1}{2}} d\theta\,,$$

because (fig. 10)

$$\varrho_1 = \mathrm{FP} = z\cos\theta + a\left(1-\frac{z^2}{b^2}\sin^2\theta\right)^{\frac{1}{2}},$$

$$\varrho_2 = \mathrm{FP}' = z\cos\theta - a\left(1-\frac{z^2}{b^2}\sin^2\theta\right)^{\frac{1}{2}},$$

$$\varrho_1 - \varrho_2 = 2\left(1-\frac{z^2}{b^2}\sin^2\theta\right)^{\frac{1}{2}}.$$

Taking the integral between the required limits $\sin\theta = \frac{b}{z}$, and $\theta = 0$ we obtain

$$\frac{4k\,\mathrm{RI}'a\cos^2 2i}{\pi b^2}(\mathrm{A}-\mathrm{H})_{\frac{z}{b}} \quad . \quad . \quad . \quad (12),$$

$(\mathrm{A}-\mathrm{H})_{\frac{z}{b}}$ denoting the difference between the asymptote and the infinite hyperbolic arc whose major axis is unity, and eccentricity $\frac{z}{b}$, a finite quantity, though A and H are severally infinite. But, if the distance from the centre to the focus be equal to unity, the transverse axis is $\frac{z}{b}$ and $(\mathrm{A}-\mathrm{H})_{\frac{z}{b}} = \frac{z}{b}(\mathrm{A}-\mathrm{H})$, which expressed in complete elliptic functions of the second order gives

$$(\mathrm{A}-\mathrm{H})_{\frac{z}{b}} = \frac{z+b}{b}\,\mathrm{E}_{e_1} - \frac{z}{b}\mathrm{E}_e\,,$$

e being put for $\frac{b}{z}$, the relation between the moduli e and e_1, being

$$e_1 = \frac{2\sqrt{e}}{1+e} = \frac{2\sqrt{bz}}{z+b}\,;$$

and e' and e'_1 denoting the corresponding quantities when b' is put for b.

Substituting in Equation 12, we get for the light reflected from the space around C,

$$\frac{4k\,\mathrm{RI}'a\cos^2 2i}{\pi b^2}\left\{\frac{z+b}{b}\mathrm{E}_{e_1} - \frac{z}{b}\mathrm{E}_e\right\}\,;$$

and similarly the light reflected from the space around D,

$$= \frac{4k\mathrm{R}'\mathrm{I}'a'\cos^2 2i}{\pi b'^2}\left\{\frac{z+b'}{b'}\mathrm{E}_{e'_1} - \frac{z}{b'}\mathrm{E}_{e'}\right\}.$$

Hence the total intensity at O is

$$u = \frac{4k\mathrm{R}\mathrm{I}'a\cos^2 2i}{\pi b^2}\left\{\frac{z+b}{b}\mathrm{E}_{e_1} - \frac{z}{b}\mathrm{E}_{e}\right\} + \frac{4k\mathrm{R}'\mathrm{I}'a'\cos^2 2i}{\pi b'^2}\left\{\frac{z+b'}{b'}\mathrm{E}_{e'_1} - \frac{z}{b'}\mathrm{E}_{e'}\right\},$$

and substituting for a, b, a', b', R, and R′

$$u = \frac{4c(v+z\cot 2i)k\mathrm{I}'\sin 4i}{\pi(v-z\tan i)}\left\{\frac{v+(2c-\tan i)z}{v-z\tan i}\mathrm{E}_{e_1} - \frac{2cz}{v-z\tan i}\mathrm{E}_{e}\right\}$$

$$+ \frac{4c(v-z\cot 2i)k\mathrm{I}'\sin 4i}{\pi(v+z\tan i)}\left\{\frac{v+(2c+\tan i)z}{v+z\tan i}\mathrm{E}_{e'_1} - \frac{2cz}{v+z\tan i}\mathrm{E}_{e'}\right\}, \quad . \qquad (13),$$

the values of the moduli in terms of c, v, z, and i being

$$e_1 = \frac{2\sqrt{2cz(v-z\tan i)}}{v+(2c-\tan i)z}, \quad e = \frac{v-z\tan i}{2cz}, \quad e'_1 = \frac{2\sqrt{2cz(v+z\tan i)}}{v+(2c+\tan i)z} \text{ and } e' = \frac{v+z\tan i}{2cz}.$$

Since the value of u will not be altered by substituting for v and z any two quantities having the same ratio, it follows that the intensity of the reflected light is uniform along the line which joins A and O (fig. 7).

ARTICLE 9.—Corollary. The value of k, the fraction which expresses the relation between the intensities of the reflected and incident rays, may be found by means of a conical reflector, thus:—

Let R = distance from the axis of a small zone described by AB (fig. 11).
r = distance from the axis at which the reflected light or heat becomes, by convergence to its axis, equal in intensity to the incident.

Then

$$k\mathrm{I}' : \mathrm{I}' :: r : \mathrm{R},$$

$$\therefore k\mathrm{R}\mathrm{I}' = r\mathrm{I}', \text{ and } k = \frac{r}{\mathrm{R}}.$$

This result, calculated on the assumption that all the rays emanating from the sun are parallel, will not deviate perceptibly from the truth, except when r is small compared with R.

The value of k may also be found by using the combination of n plane mirrors. By Article 3, Equation 2, intensity

$$= \frac{nk\mathrm{I}'\rho^2\cos i}{r^2},$$

but if the direct rays of the sun fall on the same plane with the light reflected by the combination, the intensity will be

$$\frac{nk\mathrm{I}'\varrho^2\cos i}{r'^2} + \mathrm{I}'.$$

Bringing the combination nearer the plane on which the light is thrown, the intensity of the n mirrors can be made equal to the above equation by diminishing the value of r.

Hence $$\frac{nk\mathrm{I}'\varrho^2\cos i}{r'^2} = \frac{nk\mathrm{I}'\varrho^2\cos i}{r^2} + \mathrm{I}',$$

$$\therefore k = \frac{r^2\,r'^2}{n\varrho^2(r^2 - r'^2)\cos i}\,.$$

As these results are independent of the absolute value of I′, the equality of temperature may be detected by a Fahrenheit thermometer, or any more delicate means of indicating equal temperature.

When k is accurately known, this combination, or the conical reflector, may obviously be used to ascertain the intensity of the solar beams at different hours of the day and different periods of the year, and will thereby furnish data for estimating accurately the heat or light absorbed by the atmosphere. The light lost by the solar rays in penetrating the atmosphere being known, the intensity of the radiation at different parts of the solar disc may be found by (Article 6, Equation 6), if a zone or segment of a parabolic reflector can be constructed having a focal length of 70 or 80 feet.

ARTICLE 10.—Prop. *When two Conical Mirrors have a common Axis, their Surfaces being either perpendicular or parallel, if the rays incident on the exterior Reflector parallel to the Axis meet after reflection the interior one, they will be again reflected parallel to the Axis in a beam of increased intensity.*

Let AB (figs. 12 and 13) be the common axis of two conical reflectors described by the revolution of the lines MN and CD about the axis AB, CD being either perpendicular or parallel to MN.

When AB (fig. 12) is directed to the centre of the sun, the rays which fall on the surface described by MN will make with it the angle

$$\mathrm{SKN} = \mathrm{MKH} = \mathrm{KHD} = \mathrm{FHC} = \mathrm{DCB}\,,$$

because DCB = SKN, therefore FH is parallel to AB (Euclid, 1–27). The intensity of the finally reflected beam at H is to that incident at K as the perpendicular distance of K from AB is to the perpendicular distance of H from

AB. Since every cylindrical annulus of rays incident on the exterior mirror forms after reflection from the interior a cylindrical annulus of equal thickness, the intensities must be inversely as the mean radii, the loss from reflection being neglected, or, if taken into account, equal to k^2 times the preceding intensity nearly.

ARTICLE 11.—The preceding annulus of rays may be thrown upon a circular area whose diameter is equal to the breadth of the zone, which forms a section of the annulus. To effect this, we have only so to increase the angle DCB, that the rays may meet the axis at the required distance, as shown in fig. 14.

In like manner, the parallel rays, from any extent of reflecting surface may be thrown upon the area whose section is F*f* (fig. 15) by constructing one or both of the conical mirrors of frustums having the required extent of curved surface and the requisite inclination, the same axis being common to all. But when the breadths of the annuli are small compared with the distance of the focus F*f* from the reflectors, the diameter of the circular area mentioned must be increased by the diameter of the sun's image for that distance.

Cor. When the number of lines CD, DE, EG, &c. (fig. 15), is indefinitely increased, their lengths being diminished, CDEG becomes part of a parabola.

ARTICLE 12.—Prop. *If two Parabolic Reflectors have a common Focus, the Solar Rays which are made to converge by reflection from the exterior Mirror will again form a beam of parallel Rays by reflection from the surface of the interior one.*

Let MKN and DHC (fig. 16) be sections of two confocal parabolic reflectors of which AB is the axis of the exterior and CE that of the interior, *f* being their common focus.

When AB is directed to a point in the sun's disc, the rays which fall on the exterior mirror parallel to the axis AB, in converging to *f*, will meet the surface of the interior mirror, and be reflected parallel to its axis *f*CE, as indicated by the course of the rays SKHF. Thus the solar beam of light which falls on the exterior and larger mirror is again reflected into a beam of parallel rays, and the intensity of the final beam will be greater than that of the incident, as explained in the preceding article. Moreover, since the axis *f*E of the interior mirror may make any angle with AB the axis of the exterior, the final beam may be thrown in any direction.

Article 13.—Prop. *The Rays which converge to the focus of the exterior Parabolic Mirror may be thrown by a second reflection on a given circular space, by constructing the interior reflector in the following manner :—*

Let DH and H′D (fig. 17) be parts of parabolas, whose common focus is *f*, their axes being respectively *f*C parallel to RG, and *f*C′ to QG.

By causing these to revolve about the line *f*A, a surface will be described, such that the rays converging to *f* and falling on DH, will be reflected parallel to *f*C, and fall on the plane FF′.

In like manner, the rays converging to *f* and falling on H′D, will be reflected parallel to *f*C′, and will intersect the axis of revolution at G, and fall upon the plane FF′.

If this surface be substituted for the inner reflector DHC (fig. 16), the rays reflected from the outer mirror whose axis is directed to the sun, when converging to *f*, will meet the inner reflector described by H′QDH, and be reflected (as indicated in fig. 17) so as to intersect its axis of revolution at G, and fall upon the plane FF′. And this is true, whatever angle *f*G makes with the axis of the exterior reflector.

Cor. When the number of parts in H′QDRH are indefinitely increased, and their lengths diminished, it evidently becomes the arc of a hyperbola whose foci are *f* and G.

Article 14. The convergence of the solar rays upon a given area can also be effected by combining a number of exterior reflectors, each with its corresponding interior, as indicated in fig. 16, the axes of all the exterior parabolic reflectors being directed to the centre of the sun's disc, while the axes of the interior are directed to the centre of the given spot, on which the light has to be cast. Neither is it necessary that the respective reflectors should be complete symmetrical paraboloids: the exterior may consist of a series of large plates, each forming a part of a paraboloid of revolution, with a corresponding plate cut from a less paraboloid for its inner reflector. If the axes of all the exterior plates be in the same straight line, such a combination may have a common focus, each interior having that diameter of its generating parabola, which passes through the centre of the plate directed to the spot on which the light is required to fall. The practicability of such a combination is evident, from fig. 16, where K may be viewed as the centre of one of the exterior plates, and H that of its corresponding interior, having the diameter of its generating parabola, which passes through H, directed to the plane on which the light is concentrated. The exterior plates may be joined together to move as one piece, and in like manner the interior. This combination is capable of casting the finally reflected beam in a direction making any angle with the axis of the exterior plates; and by a readjustment of the inner plates, the distance at which the rays finally meet may be varied at pleasure.

Article 15. It is stated that the Archimedean burning mirror was hexagonal. Let us consider if the combinations we have been illustrating can be made to conform to that figure. The term hexagonal may have reference either to the appearance of the mirror as a whole, or to the form of each individual reflector. Figs. 14 and 15 will correspond to the former; for if the external and internal conical frustums be each divided into six equal segments, with sufficient space between the segments to admit of free motion, the combination, viewed at a distance, will resemble a hexagonal polygon. Assuming the other meaning to be the correct one, we have only to suppose the form of the parabolic plates, which constitute the exterior and interior reflectors previously explained, to be hexagonal. From this would result two advantages:—they could be so formed that the different six-sided figures would unite together without leaving any interval, and the section of the beam cast by each on the required spot, approximating to a circle, would approach more nearly to the maximum effect with a given section of solar light.

Article 16.—Prop. *As the Expansion of the Sun's image is in proportion to the distance from the Point of Reflection, no greater accuracy is required for the construction of curved surfaces, capable of producing Combustion at distances of* 150, 200, *and* 300 *feet, than for those of a focal length of only a few inches.*

The expansion being about 1 foot in diameter for every 108 feet of focal distance, it follows that a reflector is sufficiently accurate for a burning glass, if it can concentrate the rays which fall on each part of its surface from the centre of the sun's disc, within a circular area, whose diameter is the same multiple or part of 1 foot which its focal length is of 108 feet.

The same principle may be exhibited in another and more definite form. Burning-glasses, which produce at the focus an intensity equal to parabolic ones, may be constructed of plane reflectors arranged as tangent planes to a paraboloid of revolution.

Since the increment of intensity is $\frac{k\mathrm{I}'\partial\mathrm{A}}{\text{area of circle DE}}$ (Art. 1, Equa. 6, fig. 1), if we take a portion of the surface of the paraboloid subtending an angle at the focus not greater than the sun's disc, the denominator of the foregoing fraction may be considered constant; and the intensity at the focus reflected from such an extent will be

$$\frac{k\mathrm{I}'\text{ area of circle DE}}{\text{area of circle DE}} = k\mathrm{I}'.$$

But a plane mirror which is a tangent to the paraboloid at the same spot, and of just sufficient area to reflect the circular beam of light whose section is equal to the circle DE, will also produce at the focus an intensity equal to kI' (Art. 2, Equa. 5); that is, the concentration at the focus is the same, whether a circular beam of the section mentioned be reflected from the surface of the paraboloid or from its tangent plane; and the same will evidently apply to any polygonal beam capable of being inscribed in the circular.

Hence a burning mirror, scarcely inferior in its effects to a parabolic one, may be formed of plane hexagonal reflectors, their sizes, of course, depending on the distance of the focus. For example, as the sun's image overspreads an area of 1 inch in diameter at a distance of 9 feet, a burning mirror of that focal length may be formed of plane hexagonal pieces, each side about half an inch; whereas at 108 feet distance, the sides of the plane hexagonal plates need not be less than half a foot, and so on in proportion.

Plates of the latter size being greater than those with which BUFFON performed his experiments, we infer that his combination, at distances exceeding 100 feet, would be little inferior in power to a parabolic segment of equal focal length, and capable of reflecting exactly the same sectional area of the solar beams.

Again, what has been proved true of plane mirrors, tangents to a paraboloid of revolution, must be equally true of a series of tangential circumscribing conical frustums. In all these cases, however, it is probable that the advantage in practice will remain with the parabolic figure, from the light at its focus having a greater area of maximum intensity.

From these results, as well as from independent calculations, we conclude that refracting burning-glasses may be constructed, by placing at some distance from an axis a series of acute-angled conical zones, or wedge-shaped pieces of glass (fig. 18), built up like the compound lens of BREWSTER, which will produce combustion at as great distances as BUFFON'S combination of reflectors.

ARTICLE 17. That the practibility of the Archimedean mirror may be made still more apparent, we shall now apply Equations 1 and 2, Article 3, to find the numerical intensity of the light in the focus of BUFFON'S combination. This was attempted by PEYRARD, but his conclusions are vitiated by the false premises from which he set out. He assumed that the intensity is uniform at every part of the luminous image reflected by a plane mirror,—a supposition proved incorrect by Art. 2. In our calculation we shall suppose that each of BUFFON'S mirrors, which were 8 inches by 6, produced an effect equivalent to a circular beam of 6 inches diameter, when it leaves the mirror.

Taking $k = \frac{1}{2}$, and $\cos i = 1$, we obtain by substitution, in the equations of Art. 3, the following results :—

On 23d March, heat produced by 40 glasses at 66 ft. French $= \dfrac{40 \times \frac{1}{2} \times 6^2}{\left(\frac{66}{9}\right)^2} I' = 13{\cdot}3\, I'$

Do.	do.	98	„	126 ft.	$=9\ I'$
3d April, 4 P.M.,	do.	112	„	138 ft.	$=8{\cdot}57\, I'$
10th April, after 12 noon,	do.	128	„	150 ft.	$=8{\cdot}29\, I'$
10th April, 2·30 P.M.,	do.	148	„	150 ft.	$=9{\cdot}95\, I'$
11th April, 2·30 P.M.,	do.	21	„	20 ft. $=21 \times \frac{1}{2} I' = 10{\cdot}5\, I'$ (by Art. 3, Eq. 1)	
Do. 2·30 P.M.,	do.	12	„	20 ft. $=12 \times \frac{1}{2} I' = 6\, I'$.	

In the first experiment, on the 23d March at noon, tarred beech was ignited with the 40 glasses; but the mirror not being mounted on a stand, acted at a disadvantage. On the same day, when 98 glasses ignited a plank smeared with tar and brimstone, the mirror is said to have been still more disadvantageously placed. The experiment on the 3d April was at 4 o'clock P.M., with the mirror mounted, and placed on its stand. The sun being weak, a slight inflammation was produced on a plank covered with threads of wool.

On the 10th April, with a clear sun, the 128 glasses very suddenly kindled a plank of tarred fir. At half-past 2 o'clock on the same day, the combination of 148 glasses was tried on a plank of beech tarred in part, and covered in some places with shreads of wool. The inflammation, which was very sudden, commenced on those parts of the wood which were uncovered. Beech previously charred was the material ignited with 21 glasses, and little combustible materials were the substances set on fire by 12 glasses, on the 11th April.

An inspection of the results in the preceding table shows that if $k = \frac{1}{2}$ be correct, wood done over in the manner mentioned can be ignited by a heat varying from eight to nine times that of the direct mid-day rays of the sun at Paris in April, and finely divided combustible substances by a heat considerably less, as proved by the experiment with 12 mirrors. But if we assume $k = \frac{2}{3}$, which is probably nearer the truth, the heat required to produce the same effect will vary between ten and eleven times the sun's mid-day heat. If the number in the right-hand column of the preceding table be multiplied by $\frac{1}{k}$, it gives the minimum number of plane mirrors capable in each case of producing ignition—that is, the number of mirrors which come under Equa. 1, Art. 3.

Supposing $k = \frac{1}{2}$, we find the numbers, for the first five experiments adduced, to be respectively 26·6, 18, 16·58, 17·14, and 19·18. The sizes of these mirrors will, of course, depend on the distance of the focus, and the angle at which they receive the incident light. When the distance is about 108 feet, each of them should have an extent sufficient to reflect a beam of solar light, not less than one foot in diameter, and their dimensions vary in the same ratio for other focal lengths.

As these minimum combinations have been calculated on the assumption that the centre of the luminous circle reflected by each of the mirrors can be directed with precision to a given point, which in practice is not attainable, their number, or the size of each, must be somewhat increased, to compensate for defective adjustment. After making such an allowance, it will appear that at distances not exceeding 150 feet, between 16 and 20 plane mirrors, each $2\frac{1}{2}$ by $1\frac{1}{2}$ feet, may be substituted for the numerous combinations of Buffon, the adjustment of which required so much time and trouble.

What has been shown respecting the power of the solar rays to produce combustion after one reflection, can easily be extended to the case in which the light undergoes two reflections, the intensity of the final beam being then reduced to about one-fourth that of the direct solar rays. To compensate for this diminution, the exterior reflectors must have about twice the area of Buffon's combination. Taking, for example, the 128 plane mirrors which kindled combustibles at 150 feet, the reflecting surface of the combination is $\frac{8 \times 6 \times 128}{144} = 42\frac{2}{3}$ square feet, and the sectional area of solar light, which we supposed it to reflect, was $\frac{\pi}{4} \times 6^2 \times 128 \div 144 = 25$ square feet nearly. It seems then, that about 80 feet of reflecting surface, or an extent capable of reflecting 50 square feet of solar light, will be more than sufficient to inflame such a material as tarred wood, &c., at the distance of 150 feet, after having undergone two reflections,—an extent of surface not too great to be united in one compound mirror, constructed in the manner explained in the foregoing articles.

Considering the scepticism which has prevailed respecting the Archimedean achievement in the most favourable circumstances, we are the less surprised to find that some recent authors, in quoting the passage from Tzetzes, omit the statement which refers to the burning of the Roman ships in *winter*. Instead of ignoring these winter attacks, let us examine them in the light which Buffon's experiments supply. That performed with 112 mirrors, at the distance of 138 feet, was at 4 o'clock on the afternoon of the 3d April, at which time the altitude of the sun in the sky of Paris would nearly correspond to his meridian altitude at mid-winter in the more southern latitude of Syracuse; and as the difference between the meridian altitude of the sun at the summer and winter solstice amounts to above 46°, it must be admitted that this additional fact corroborates in a striking manner the evidence already adduced.

Having now shown how compound burning mirrors can be constructed corresponding in every respect to the description which Tzetzes gives of the one invented by Archimedes, and that every statement in the passage is in accordance with well-established facts, we conclude that his narrative is no fiction, but, on the contrary, a true account of a real mirror, capable of producing all the effects ascribed to it. While this ancient discovery can be tried, after the lapse of two thousand years, by the light of modern science, and pass with credit through the ordeal, the pretended discoveries of comparatively modern times, when subjected to the same test, fall to pieces.

An instance may be given without digressing from the subject of our paper. Thomas Digges, who republished in 1591 a work by his father, Leonhard Digges, entitled "Pantometria," would make us believe, in the preface to this edition, that he had seen his father at sundry times fire gunpowder and discharge ordnance at a distance of half-a-mile or more, by means of the sun's beams. Had he been aware that to accomplish such a feat would require at least four thousand square feet of reflecting surface, we may venture to affirm that he would not have overstepped so far the Archimedean range.

We may observe, in conclusion, that the experiments of Buffon, taken in connection with the preceding deductions, are calculated to produce a strong conviction that, in clear and comparatively warm climates, the sun's rays may be made, at a small expense, to supersede in some respects the fires employed in culinary operations. Further, when it is considered with what ease a combination of plane mirrors, or a series of conical reflecting zones, can be constructed, capable of producing a heat exceeding that of the most intense furnace (Art. 16), we infer that the solar beams may also be turned to account by the chemist and metallurgist. For these purposes, one reflection only is required, as the reflected light can be made to fall always on the same spot, by directing the axis of the reflector to the centre of the sun's disc, and causing it to follow the sun's motion in the heavens, by revolving round a fixed axis parallel to that of the earth.

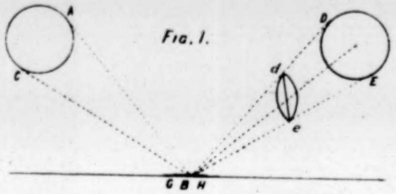


Fig. 1.

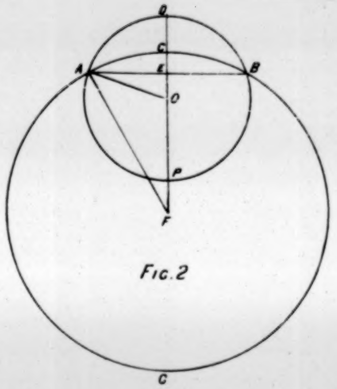


Fig. 2

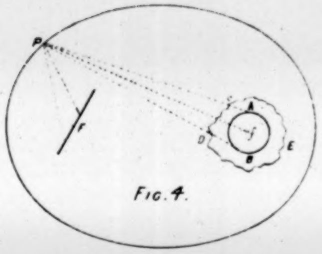


Fig. 4.

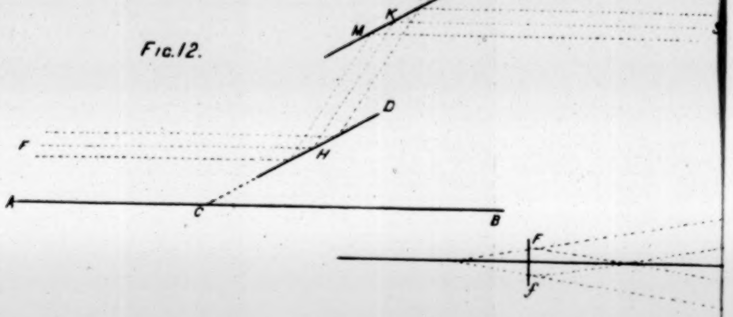


Fig. 12.

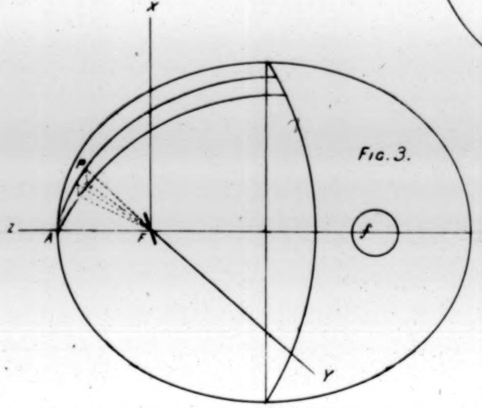


Fig. 3.

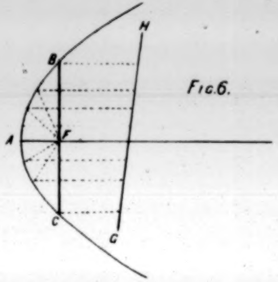


Fig. 6.

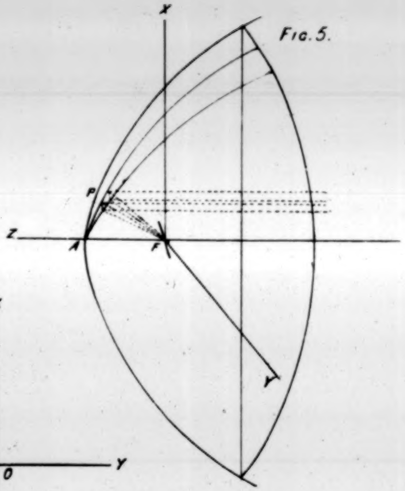


Fig. 5.

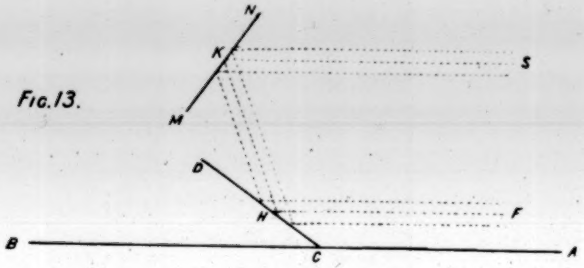


Fig. 13.

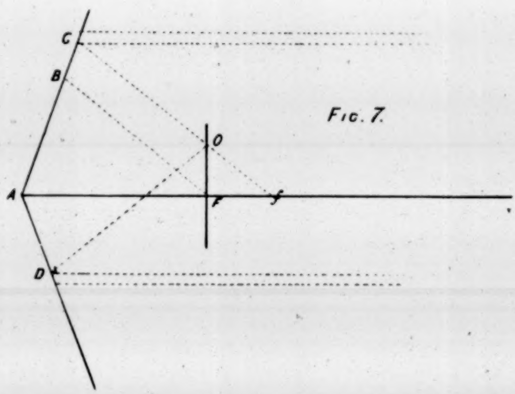


Fig. 7.

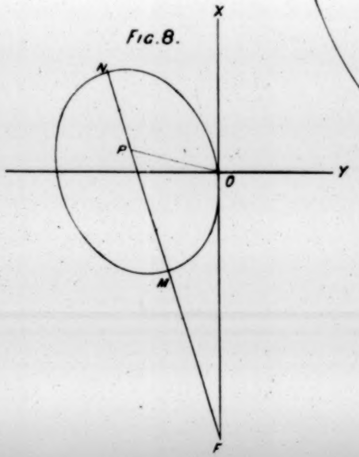
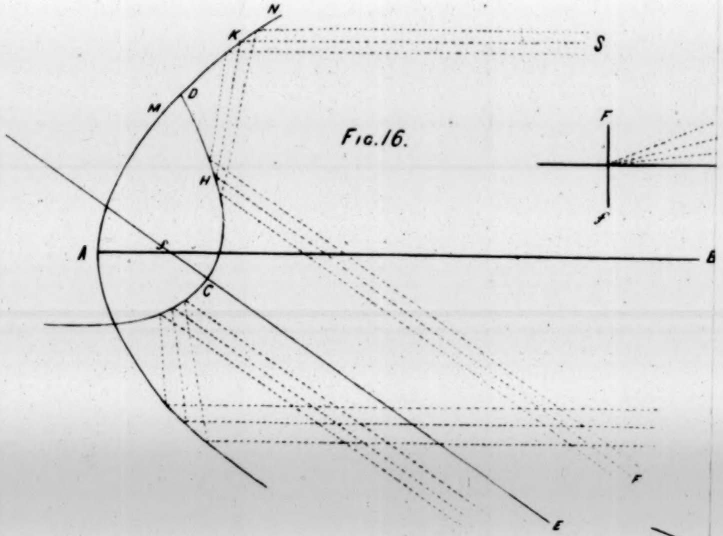


Fig. 8.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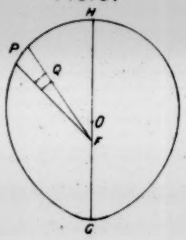
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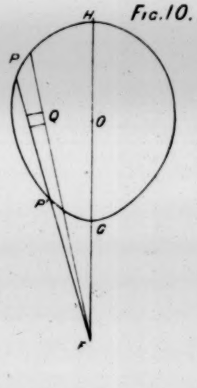
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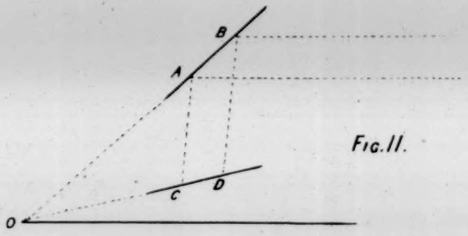
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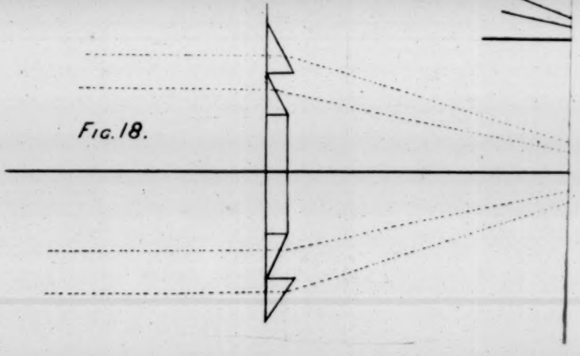
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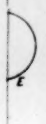

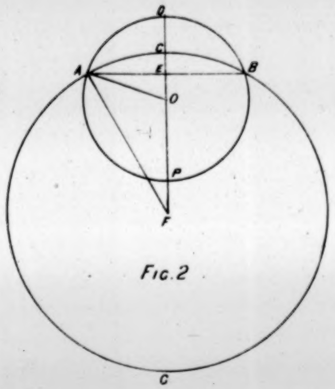

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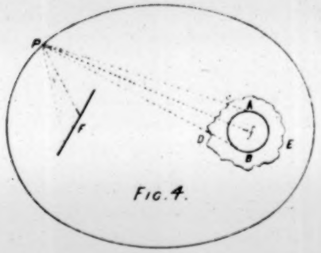

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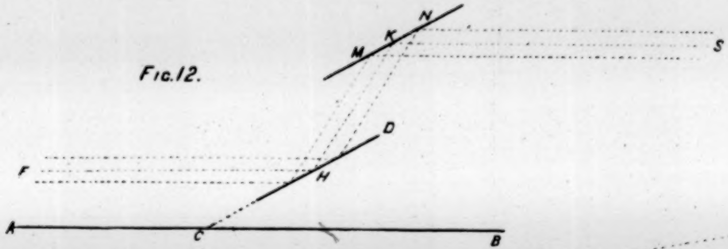

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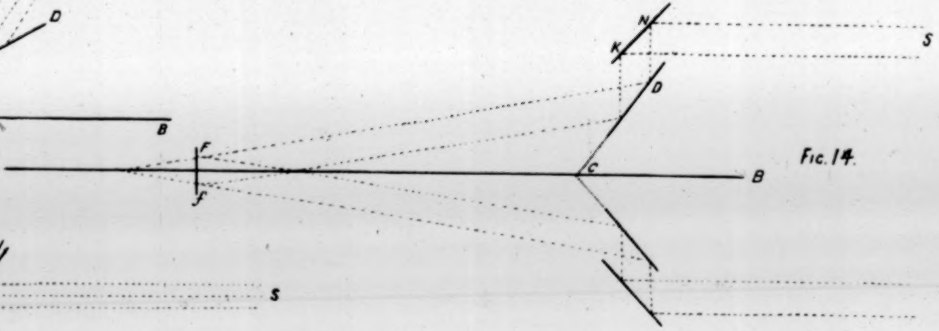

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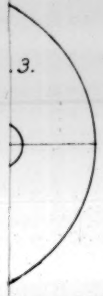

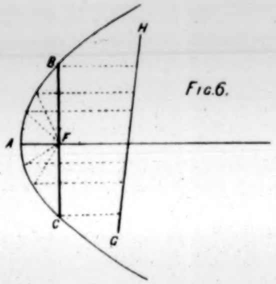

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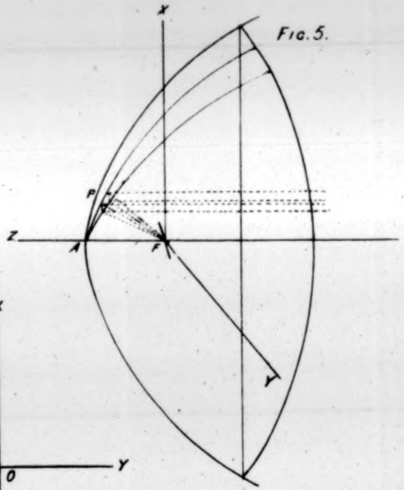

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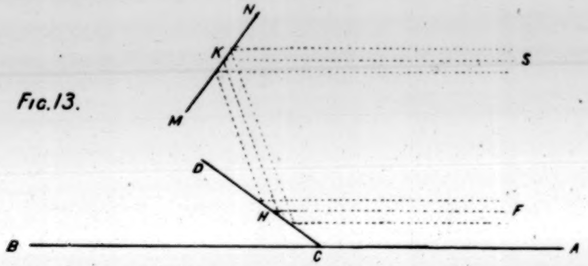

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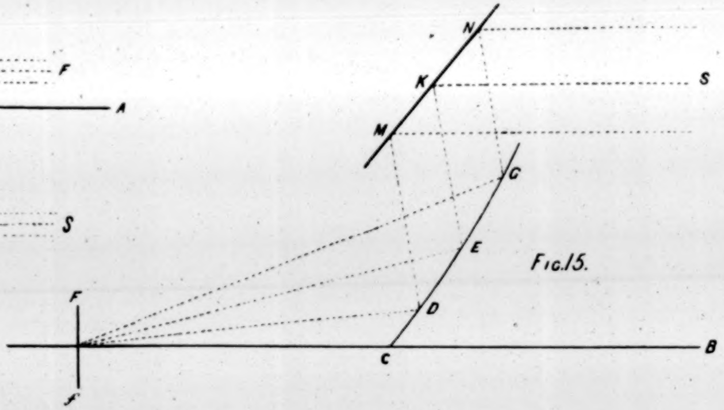

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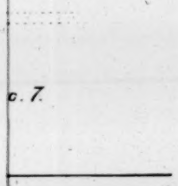

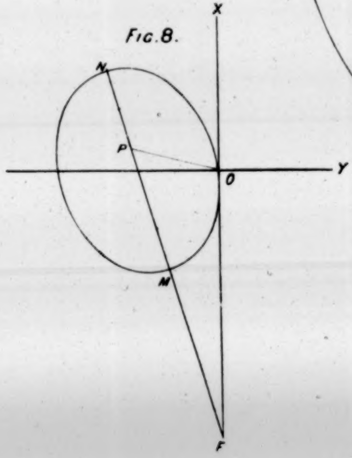

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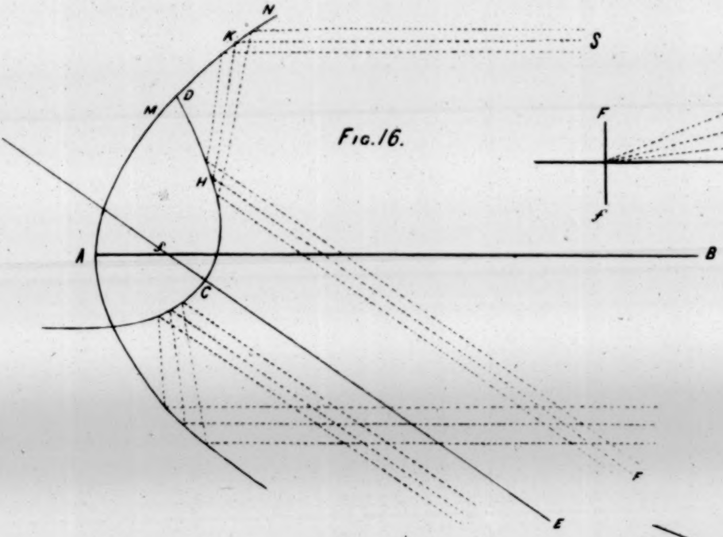

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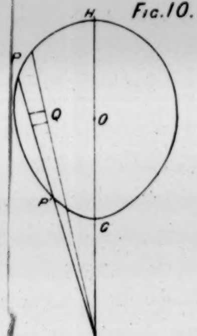

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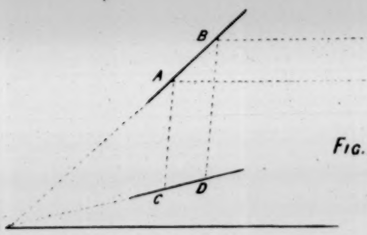

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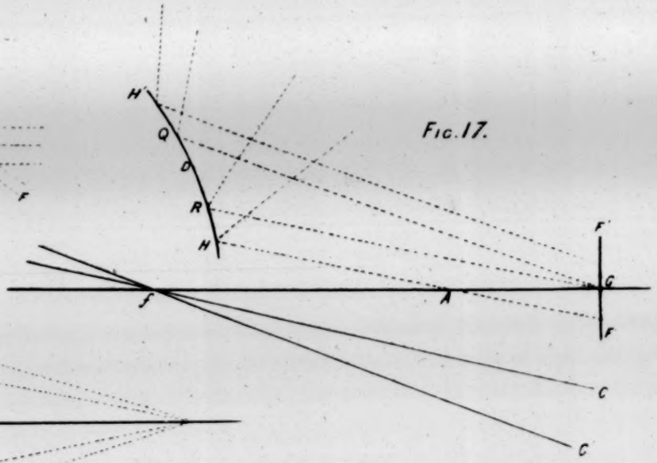

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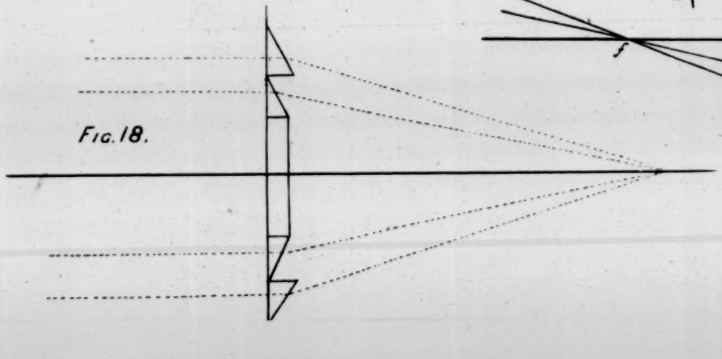

Fig. 18.

V.—*On the Connection between Chemical Constitution and Physiological Action.* Part. I.—*On the Physiological Action of the Salts of the Ammonium Bases, derived from Strychnia, Brucia, Thebaia, Codeia, Morphia, and Nicotia.* By Dr A. Crum Brown and Dr Thomas R. Fraser.

(Read 6th January 1868, under the title "On the Changes produced by direct Chemical Addition on the Physiological Action of certain Poisons.")

There can be no reasonable doubt that a relation exists between the physiological action of a substance and its chemical composition and constitution, understanding by the latter term the mutual relations of the atoms in the substance. There are numerous indications of such a relation, and attempts have been made to express it formally in certain cases. Thus it has been long observed, that the salts of the same base have a common physiological action, and it has been pointed out by Mr Blake* that, with some exceptions, the salts of isomorphous bases have a similar action. A corresponding likeness in physiological action may be traced in salts having the same acid, but beyond these generalisations we are not aware that any approach has been made to the statement of a law connecting the physiological action of a substance with its chemical constitution.

Some observers have endeavoured to connect physiological action with composition, looking for the cause of the peculiar action of substances in the presence or proportion of particular elements. It is a sufficient answer to this to point to isomeric or polymeric bodies—bodies having identically the same composition—which differ totally in action, such as acetic acid ($C_2H_4O_2$), and sugar ($C_6H_{12}O_6$); glycocoll ($C_2H_5NO_2$), and nitrite of ethyl ($C_2H_5NO_2$); or to instance kakodylic acid, which is inert, although perfectly soluble, and containing more than 54 per cent. of metallic arsenic.

Examples such as these clearly show that composition alone is quite insufficient to explain physiological action, and that constitution must also be taken into account in every attempt to connect the chemistry of substances with their action on the animal body.

The most direct way of making such an attempt would obviously be to compare physiological action and chemical constitution in a sufficiently large number of cases, and by classifying the results to deduce a law; but, unfortunately, the data which we possess are quite insufficient for this. We know, indeed, the "structure" of a considerable number of substances; that is, we know the *order* in which the atoms of these substances are related to each other, but something more than this is implied in the term *constitution*, as we have used it above. For this involves not only the "structure," or the arrangement of the equivalents in atoms and in mutually united pairs, but also what we may call the *potential* of each pair of united equivalents.[*] For instance, the structural formula of formic acid is

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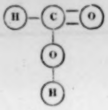
* Proceedings of the Royal Society of London, vol. iv. Jan. 28, 1841, p. 285.



which indicates—1*st*, That the four carbon equivalents form one atom, the four oxygen equivalents two atoms, and the two hydrogen equivalents two atoms; 2*d*, that these equivalents are united in pairs, thus—*co*, *co*, *co*, *ch*, *ho*, but it does not in any way indicate (and we do not know) what is the *potential* of each of these pairs—that is, how much energy would be required to separate the equivalents from each other. We know that this potential depends upon the structure, and we can to a certain extent trace the nature of this dependence, but we cannot as yet express the potential numerically, or give a rule for finding its value from the structure, and till we can do this we do not fully know the constitution.

But even the structure of the majority of substances is not at all, or only very imperfectly known, and this is especially the case with those whose physiological action has been most fully investigated, such as the natural alkaloids.

Seeing, then, that we could not follow the direct road of induction, it occurred to us that a by-path might be found, by making use of a method resembling in its main features a mathematical *calculus of finite variations*. This method consists in performing upon a substance a chemical operation which shall introduce a known change into its constitution, and then examining and comparing the physiological action of the substance before and after the change. We may express this in mathematical language thus:—Let C represent the constitution of the original substance and Φ its physiological action. After the operation, C becomes $C + \Delta C$ and Φ, $\Phi + \Delta\Phi$. Here ΔC, Φ, and $\Phi + \Delta\Phi$ are known, and by applying the method to a sufficient number of substances, and by varying ΔC, we might hope to determine what function Φ is of C. The only reason why this method is not a strictly mathematical one is, that we cannot express our known terms ΔC, Φ, and $\Phi + \Delta\Phi$ with sufficient definiteness to make them the subjects of calculation. But although, on this account, we cannot obtain an accurate mathematical definition of f in the equation $\Phi = fC$, we may be able, in an approximate manner, to discover the nature of the relation.

In applying this method, we must select a chemical operation which satisfies the following conditions:—*1st*, That it is unambiguous; that is, that the change of structure produced by it is susceptible of only one interpretation. *2d*, That the change of *structure* produced by the operation is, in all cases investigated, the same, and the change of *constitution* (ΔC)—that is, the change of structure and *potential*—as nearly as possible the same. *3d*, That the operation is completely under our control, so that it cannot be either performed or reversed spontaneously, in ordinary circumstances, within the animal body. *4th*, That the substance is equally suitable for absorption into the system before and after the change (that is, that Φ and $\Phi + \Delta\Phi$ are observed under similar conditions); and *5th*, That a decided change of physiological action is, in some cases at least, produced (that is, that $\Delta\Phi$ is not always $= 0$).

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[*] More correctly, "the exhaustion of the potential energy" of each pair of united equivalents. See THOMSON and TAIT's Treatise in Natural Philosophy, § 547.

Chemical operations may be divided into two classes—*1st*, operations of substitution; and *2d*, operations of addition or subtraction. In the first, an atom or group of atoms is replaced by an *equivalent* atom or group of atoms, without any change taking place in the active atomicity of any atom or radical in the substance.

In the case of addition (and by subtraction we mean to express merely the inverse operation to addition), the active atomicity of one or more atoms or radicals in the compound is increased, and the bonds thus set free, or rendered active, are saturated by atoms or radicals (the sum of whose active atomicity is of course an even number), which are thus *added* to the substance. We shall apply the name *condensation* to capability of being added to in whatever way the addition takes place, and distinguish two kinds of condensation, *intra-atomic* and *inter-atomic;* in the first of which it is an atom, and in the second a compound radical, the active atomicity of which is increased. Thus, carbonic oxide, sulphide of methyl, and protochloride of tin, are examples of *intra-atomic* condensation; olefiant gas, the dibasic anhydrous acids, and allylic alcohol, of *inter-atomic* condensation; while hydrocyanic acid (if we assume for it the formula $\textcircled{\scriptsize H}-\textcircled{\scriptsize N}=\textcircled{\scriptsize C}$) shows both.

Many operations of addition and also of substitution satisfy the 1st, 2d. 3d, and 4th of the five conditions mentioned above; but when we examine them in reference to the 5th condition, we find a marked difference. Operations of substitution (satisfying the 1st, 2d, 3d, and 4th conditions) do not appear greatly to change the physiological activity of a substance, except, 1st, where the activity depends on direct local action; or 2d, where the operation removes or introduces an atom or radical, the compounds of which are as a rule active. As examples of the first exception, we may take sulphuric acid (H_2SO_4) and caustic soda ($HNaO$), both poisonous; while sulphate of soda (Na_2SO_4) and water (H_2O) are not: as examples of the second, acetate of lead and cyanide of sodium, both poisonous, acetate of potash and chloride of sodium not. Besides the exceptions which can be reduced to the two classes just mentioned, there are several isolated cases of change of activity produced by replacement, such as the singular inertness of ferrocyanide of potassium and of the analogous double cyanides, as compared with the activity of cyanide of potassium and its analogues.

On the other hand, operations of addition, particularly where the condensation diminished by the addition is intra-atomic, seem, in many cases, to produce very decided change both in the kind and in the degree of the physiological activity of the substance acted on. The following examples will illustrate this statement.

Some are cases of direct and some of indirect addition, and in all of them the change of structure produced is known, and there is in none of them much risk of fallacy arising from the change taking place spontaneously in the animal system. The first column contains the names and formulæ of the substances before addition, the second the atoms or groups added, and the third the names and formulæ of the substances produced.

I.	II.	III.
Carbonic oxide, CO	O	Carbonic acid, CO_2
Hydrocyanic acid, HCN	$2H_2 + HCl$	Hydrochlorate of methylamine, CNH_6Cl
Arsenious acid, As_2O_3, $[HAsO_2]$	$(CH_3)_2$	Kakodylic acid, $AsC_2H_7O_2$ [*]
Strychnia, $C_{21}H_{22}N_2O_2$	$(CH_3(HO))$	Methyl-strychnia (hydrate), $C_{22}H_{26}N_2O_3$ [†]
Brucia, $C_{23}H_{26}N_2O_4$	$(CH_3(HO))$	Methyl-brucia (hydrate), $C_{24}H_{30}N_2O_5$ [‡]

It will be observed that all the substances in the first column are highly poisonous, while those in the third column are either quite inert, or possess an action entirely different in kind from that of the bodies from which they are derived, and very much less in degree.

A consideration of the hitherto isolated facts collected in the above table leads not unnaturally to a suspicion that *condensation* (and in particular *intra-atomic condensation*) is in some way connected with physiological activity, as the first is, and the second appears to be, diminished or removed by chemical addition. This suspicion is strengthened when we observe that in a very large proportion of the cases as yet investigated saturated bodies (that is, bodies whose condensation is 0) are inert, or nearly so.

Kakodylic acid, as already mentioned, is a remarkable example of this, and the salts of tetrethyl-arsonium[§] seem to be equally inert. Similarly, the salts of tetramethyl-stibonium[‖] are not emetic. So that, as far as experiment goes, it would seem that the stable compounds of pentatomic arsenic and antimony have a very different and much less strongly marked action than the compounds in which these elements are contained as triads, or than those (such as arsenic acid) in which, although present as pentads, they are easily reduced by subtraction to the state of triads.

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[*] BUNSEN, Annalen der Chemie and Pharmacie, vol. xlvi. p. 10 (1843).

[†] STAHLSCHMIDT, Poggendorff's Annalen, vol. cviii. p. 523 (1859).

[‡] Ibid. p. 541.

[§] LANDOLT, Annalen der Chemie und Pharmacie, vol. lxxxix. p. 331 (1854).

[‖] Ibid. vol. lxxxiv. p. 49 (1852).

In reference to this, we cannot avoid referring to a very remarkable passage in BUNSEN's admirable paper on kakodylic acid. After describing the experiments by which he proved the inert character of this acid, he says, "Gehen wir auf den Grund dieser unerwarteten Erscheinung zurück, so bietet sich dafür nur in der Annahme eine Erklärung dar, dass die Verbindungsweise des Arseniks im Kakodyl eine andere ist, als in seinen unorganischen Verbindungen. Indem es darin aufgehört hat, für sich einen Angriffspunkt der Verwandtschaft zu bilden, hat es zugleich seine Reaction auf den Organismus verloren." (Annalen, vol. xlvi. 1843, p. 11.) While it is plain that BUNSEN does not here refer to the different degree of saturation of the arsenic in arsenious and kakodylic acids, both because the whole theory of saturation is of a much later date, and because he makes no distinction between the mode of combination of the arsenic in those compounds in which kakodyl is monad and arsenic triad, and those in which kakodyl is triad and arsenic pentad, he points out in an exceedingly clear manner the striking coincidence of peculiar chemical constitution and peculiar physiological action in the case of kakodylic acid.

While, however, the cases mentioned incline us to believe that physiological activity is related to condensation, the occurrence of saturated substances, such as alcohol, corrosive sublimate, and oxalic acid, having a well marked poisonous action, and of condensed substances, such as benzoic acid and salicine, which are comparatively inert, shows that condensation is not the only condition of physiological activity. There can, at the same time, be little doubt that if the effect of condensation were discovered and eliminated, the other conditions might be much more hopefully sought for.

Under these circumstances, we turned our attention, in the first place, to the effect of chemical addition in altering the physiological action of the natural alkaloids. We were led to do so, partly by a consideration of the ease with which, by means of iodide of methyl, the nitrogen of nitrile bases can be rendered stably pentatomic, and partly by the hope, grounded on the observations of STAHLSCHMIDT in reference to the salts methyl-strychnium and methyl-brucium, that we should obtain marked changes of physiological action.

The great majority of natural alkaloids belong to the class of *nitrile bases*, that is, they contain one or more atoms of triatomic nitrogen directly united to carbon by three bonds. This nitrogen atom (or, in the case of poly-acid bases, atoms) can become pentatomic, as in the formation of salts; thus in the formation of hydrochlorate of morphia the nitrogen takes up H and Cl, thus becoming pentatomic, united by three bonds to carbon, by one to hydrogen, and by one to chlorine. But by this change it is not rendered permanently or stably pentatomic; it easily loses the hydrogen and chlorine it has acquired, and returns to the triatomic state. The action of alkalies, or, in many cases, even of alkaline carbonates, is sufficient to effect this, and reprecipitate the alkaloid. It is obvious, therefore, that the chemical addition of an acid does not satisfy the third condition mentioned above, for it is certain that the addition can be performed in the stomach, which is acid, and very probable that it may be reversed in the blood and other alkaline fluids of the body. But if, instead of an acid, we make use of such a substance as iodide of methyl, we find that while the triatomic nitrogen takes up CH_3 and I, and becomes pentatomic (just as in the former case it took up H and Cl), it does not lose these newly-acquired atoms when the substance is treated with alkalies, but remains pentatomic even when subjected to attacks more violent than any to which it can be exposed in the animal system. This operation, the addition of iodide of methyl to nitrile bases, satisfies the first condition, for we know precisely what change of structure is produced. It satisfies the second, for the change of structure is the same in all nitrile bases; and the change of *potential*, as far as can be judged from a very rough estimate of the heat produced by the change, and from the general character of the substances produced, is not very different in different cases. It satisfies the third, as we have seen above; and as the iodides of the compound ammoniums thus formed from the alkaloids are all tolerably soluble in warm water, and can easily be transformed into other salts very readily soluble, it satisfies the fourth condition; and the observations of STAHLSCHMIDT show, and the sequel of this paper will further prove, that it satisfies the fifth.

It deserves to be noted that this operation only removes the condensation of the typical nitrogen (that is, of one atom of nitrogen for each molecule of a mono-basic acid that the alkaloid can saturate), and leaves any other condensation which may exist in the substance unaffected; so that even if physiological action should depend upon condensation, it would be unreasonable to expect $\phi + \Delta\phi$ to be in all cases zero, that is, that the new bodies should be quite inert.

In the present paper we communicate the results of the application of the method described to strychnia, brucia, thebaia, codeia, morphia, and nicotia. In each case we shall first describe the action of the alkaloid itself, then give the method of preparing the derived substances, and describe their physical characters, and, with some detail, their physiological action. Our investigation of the physiological action of these substances has been chiefly directed to the determination of their poisonous activity, and of the most prominent differences between the nature of their action and that of the alkaloids from which they are derived.

STRYCHNIA.

It is well known that strychnia acts on the living economy in a distinctly defined and characteristic manner, and that it is one of the most active of poisons. When administered subcutaneously, doses varying from one-twentieth to one-fiftieth of a grain rapidly produce in rabbits the most violent tetanic convulsions, and in a few minutes kill the animal. Few poisons have been more carefully studied, and it is now almost undoubtedly established that the phenomena produced by strychnia are due to a localisation of its action on the spinal cord.

Iodide of methyl-strychnium.—Strychnia ($C_{21}H_{22}N_2O_2$) is a mono-acid nitrile base, that is, it contains one atom of nitrogen united by three bonds to carbon; the structure of the radical or radicals $(C_{21}H_{22}NO_2)'''$ is unknown. How first demonstrated that strychnia is a nitrile base by subjecting it to the action of iodide of ethyl, and described, in a paper read before this Society,[*] the ethyl-strychnium and amyl-strychnium compounds. STAHLSCHMIDT subsequently prepared and described the compounds of methyl-strychnium.[†] We prepared the iodide of methyl-strychnium by STAHLSCHMIDT'S method. Strychnia, in fine powder, was treated, in a flask, with excess of pure iodide of methyl;[‡] the flask was allowed to stand in the cold for some hours, then heated in the water-bath, the excess of iodide of methyl distilled off, and the iodide of methyl-strychnium dissolved in boiling water, filtered, and recrystallised.

Iodide of methyl-strychnium ($C_{21}H_{22}N_2O_2CH_3I$) crystallises in brilliant white scales, tastes distinctly bitter, though not so strongly or persistently so as strychnia, and when treated with strong sulphuric acid and peroxide of manganese, or bichromate of potash, it gives the colour reaction of strychnia, somewhat obscured by the presence of free iodine. It dissolves in 133 parts of water at 37° C., and in 385 parts of water at 9° C.

STAHLSCHMIDT has published a statement to the effect that the methyl-strychnium compounds are inert. As the sequel will show, we do not confirm this assertion; but it is proper to admit that our investigation arose principally from it.

We first examined the effects of this substance by subcutaneous administration. For this purpose, it was reduced to the form of very fine powder, suspended and dissolved in warm distilled water, and injected into a previously formed cavity in the subcutaneous cellular tissue. In this way, by a series of progressively increasing doses, it was found that as much as twelve grains could be given to a rabbit, weighing three pounds and four ounces, without any effect whatever. Fifteen grains, however, produced serious symptoms, though followed by recovery, and death was caused by the exhibition of twenty grains. Short abstracts of the majority of the experiments will be found in the table at the end of this paper; we shall, however, give some details of several experiments, in order to illustrate the mode of action.

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[*] Transactions, vol. xxi. p. 32 (1854).

[†] POGGENDORFF'S Annalen, vol. cviii. p. 513 (1859).

[‡] As iodide of methyl prepared directly from pyroxylic spirit is apt to become acid, it is advisable, if such impure iodide of methyl be used, to add a small quantity of an alkali (such as carbonate of potash), in order to prevent any of the strychnia being converted into a salt, and thus remaining unacted on by the iodide of methyl.

Experiment VII.—Two very small incisions were made through the skin, one in either flank, of a rabbit, weighing three pounds and eight ounces; and by inserting an aneurism needle into these incisions, two cavities were formed in the cellular tissue. Into each of these we injected seven and a-half grains of iodide of methyl-strychnium (in all fifteen grains), suspended and dissolved in warm distilled water. No effect was caused until forty-five minutes, when the rabbit moved about uneasily, the limbs gradually yielded, and it soon lay on its chin and abdomen. When placed on the side, it remained quiet, without any efforts to recover a normal posture. Irritation did not cause any spasm nor give the slightest evidence of any increase in the reflex excitability. In one hour, when lifted by the ears, it hung in a perfectly flaccid and unresisting condition; the respirations were sixty-four per minute; and there were no voluntary movements. In one hour and thirteen minutes, a few spontaneous movements occurred in the limbs, but these, apparently, were merely feeble efforts to change its position. The external temperature appeared to be somewhat elevated, and the respirations were sixty-five per minute. In an hour and twenty-two minutes, a few twitches of the body, and especially of the abdominal muscles, occurred during the respiratory movements, which were now at the rate of sixty-six per minute; the eyelids did not contract when the conjunctiva or cornea was touched; but the animal was still conscious. In two hours, the condition was nearly the same as at last note, except that faint twitches of the eyelids could be excited by gentle irritation of their edges. In two hours and fifteen minutes, a number of very feeble spasmodic-like movements of the limbs occurred along with the twitches of the body, and these could also be excited by irritation. In two hours and thirty-five minutes, the condition of the rabbit had greatly improved. Efforts to rise were frequently made, in the intervals between which it lay perfectly quiet and flaccid, and the sensibility of the conjunctiva and cornea appeared to be normal.

The observations were now stopped until the following morning, when the rabbit was found jumping actively about, and apparently in a perfectly normal condition.

Experiment VIII.—We injected ten grains of iodide of methyl-strychnium, suspended and dissolved in warm distilled water, into each of two subcutaneous cavities (twenty grains in all) of a rabbit, weighing three pounds and two and a-half ounces. Fifty minutes afterwards, the animal was lying flaccid, and exhibited the continuance of life only by slow and laboured respiratory movements. In one hour, tremulous movements of the body and limbs accompanied the respirations; and it was extremely difficult to excite even a feeble reflex movement by pretty strong stimulation. In one hour and ten minutes, the rabbit was dead.

The autopsy was immediately made: the heart was contracting with regularity and considerable force, at the rate of 160 beats per minute; the intestinal peristalsis seemed normal; galvanic stimulation of the exposed muscles caused energetic contractions, and continued to do so until more than thirty minutes after death; and similar stimulation of the exposed sciatic nerves caused contractions of the posterior extremities at four minutes after death, but ceased to do so in other five minutes.

These experiments are sufficient to illustrate the physiological effects that are produced when iodide of methyl-strychnium is administered to rabbits by subcutaneous injection. We have made similar experiments, with exactly analogous results, on dogs and cats, the more important details of which are mentioned in the table at the end of this paper.

The effects of internal administration were examined by passing a gum-elastic catheter down the œsophagus of a rabbit, and so injecting iodide of methyl-strychnium, suspended and dissolved in warm distilled water. It is unnecessary to give any description of these experiments, at this place, as no effect was produced by this method of exhibition, although as much as thirty grains was given at one time, and it was inconvenient, as well as unnecessary, to give larger doses. It is well known that to produce symptoms with a poison in a rabbit, a much larger quantity is required when it is administered by the stomach than when it is injected subcutaneously. The contrast between the action of iodide of methyl-strychnium and strychnia itself was, however, well shown in the rabbit to which thirty grains of the former had been given without any effect; for one-tenth of a grain of strychnia, also administered by the stomach, quickly produced violent tetanic convulsions, and, in a few minutes, killed the animal.

As iodide of methyl-strychnium is a sparingly soluble substance, it appeared proper, in conformity with our fourth condition, and in order to compare the actions of strychnia and of methyl-strychnium, that the properties of the sulphate of the latter, which is extremely soluble, should be examined.

Sulphate of methyl-strychnium ($(C_{21}H_{22}N_2O_2CH_3)_2SO_4$) was prepared by precipitating a hot aqueous solution of the iodide by a hot solution of sulphate of silver, the slight excess of the latter was precipitated by chloride of sodium, the filtrate evaporated to dryness, and the sulphate of methyl-strychnium extracted by means of alcohol. It crystallises in delicate white needles, is very soluble in cold water, tastes like the iodide, and gives the usual strychnia-reaction with oxidising agents.

As had been anticipated, it is much more active than the iodide. One grain, dissolved in water, and injected under the skin of a small rabbit, caused its death in eighteen minutes. Half-a-grain, however, produced no effect. When eight-tenths of a grain was similarly administered, the following symptoms were produced, but death did not result.

Experiment XXIII.—Eight-tenths of a grain of sulphate of methyl-strychnium, dissolved in a few minims of distilled water, was injected into the subcutaneous tissue over the abdomen of a rabbit, weighing three pounds and three and a-half ounces. It caused no immediate uneasiness, and the animal was unaffected for about twenty-five minutes, after which, however, it became restless. In twenty-eight minutes, movements of the limbs were made with obvious difficulty, and the rabbit occasionally stumbled. In twenty-nine minutes, the limbs could no longer support the body, and a position was assumed in which the rabbit lay on the abdomen with the chin resting on the table. It was now perfectly flaccid, and remained on the side when so placed. There was no evidence of exaggeration in the reflex motor function; indeed, an extremely violent stimulus was required to produce even a faint reflex movement. In thirty-two minutes, slight quiverings occurred, and the respirations were laboured, and at the rate of sixty-eight per minute. This condition continued until one hour after the administration, and during all this time consciousness seemed unaffected, and sensibility was not lost, as was shown by stimulation of the conjunctiva or cornea causing movements of the eyelids. Repeated efforts were, however, now made to recover a normal posture, and the frequency of the respirations increased. In one hour and eleven minutes, the head was raised from the table; and in eleven minutes afterwards, the rabbit succeeded in rising on its feet and maintained itself thus, though at first somewhat unsteadily. In one hour and twenty-two minutes, all the symptoms had disappeared. The rabbit was perfectly well on the following morning.

The sequence of symptoms to a fatal termination, and the *post mortem* appearances, are well shown in the experiment where one grain was exhibited (Experiment XXV.).

Experiment XXV.—We dissolved one grain of sulphate of methyl-strychnium in fifteen minims of distilled water, and injected this solution into the subcutaneous tissue of a rabbit, weighing two pounds and fourteen ounces. In eleven minutes, the first symptom, unsteadiness, appeared. In twelve minutes, the rabbit was lying on the abdomen and chest, with the lower jaw resting on the table. There were no voluntary movements; strong irritation caused feeble reflex movements only, and the respirations were shallow and laboured, and at the rate of sixty per minute. In sixteen minutes, quivering movements of the chest and abdominal muscles occurred, from which it was nearly impossible to distinguish the respiratory movements; and the sensibility of the eyeball was greatly impaired. In seventeen minutes, there were no movements, except occasional faint twitches of the muscles of the body, while irritation of the skin or of the eyeball did not cause any reflex movements. The rabbit was quite dead in eighteen minutes.

Four minutes after death, the heart was contracting in proper rhythm and with regularity, at the rate of 164 beats per minute, and the intestinal peristalsis was well marked; the heart had however ceased to contract in other twenty-four minutes, but the intestinal peristalsis continued for some time after this. Six minutes after death, the gluteal muscles were exposed, and exposure caused them to twitch. The sciatic nerves were at the same time stimulated with galvanism and mechanical irritation, but no contractions were produced. *Rigor mortis* commenced about two hours and forty minutes after death.

When sulphate of methyl-strychnium is administered to rabbits by the stomach, twenty-five grains appears to be about the minimum fatal dose. The symptoms and mode of death are the same as those that result from subcutaneous injection.

These experiments clearly prove that the methyl derivatives of strychnia possess a very different action from strychnia itself. In none of our experiments, not even in the fatal cases, were the symptoms those of strychnia-poisoning; no starts nor spasms occurred, nor did stimulation give evidence of the slightest increase of reflex excitability. In fact, a condition exactly the reverse of that produced by strychnia was produced by these compounds. In place of violent spasmodic contractions and muscular rigidity, the appearances were those of paralysis, with a perfectly flaccid condition of the muscles. The limbs of the animal first yielded, its head gradually sank until it rested on the table, by-and-by, it lay in a perfectly relaxed condition, and when death occurred, it was due to stoppage of the respiratory movements. In the autopsis, further evidence was obtained to distinguish the effects of the methyl-strychnium compounds from those of strychnia. The heart was found acting with nearly its normal rapidity; the spinal motor nerves were either paralysed or nearly so; and, in place of the almost immediate occurrence of *rigor mortis* that follows the action of strychnia, the muscles continued flaccid, contractile, and alkaline for many hours.

These symptoms are sufficient to suggest a close resemblance between the action of the methyl derivatives of strychnia and that of curare (*wourali*), a well known and elaborately studied poison. In a recent publication, Professor SCHROFF, of Vienna, has indicated a resemblance of this kind between the nitrate of methyl-strychnium and curare.* Both substances undoubtedly produce a condition of general paralysis; but the special characteristic of curare-poisoning is, that this paralysis is the result of an impairment or destruction of the function of the peripheral terminations (end-organs) of the motor nerves. It is impossible to demonstrate such an action without undertaking experiments of a special character. We, accordingly, extended our research for the purpose of examining this question.

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* Wochenblatt der Zeitschrift der k. k. Gesellschaft der Aertze in Wien; vi. Band, 1866, pp. 157–162.

Experiment XXVIII.—The sciatic artery and vein were tied at the knee of a frog, and one-tenth of a grain of sulphate of methyl-strychnium, dissolved in distilled water, was injected under the skin of the back. Eight minutes afterwards, the frog was lying in a perfectly flaccid state, and, in ten minutes, irritation of any portion of the skin produced energetic movements of the tied limb, *below the points of ligature*, but nowhere else. The sciatic nerve of the untied limb was now exposed, and on stimulating it with a weak, interrupted galvanic current, movements occurred in the tied limb only; not the slightest effect occurred in any part to which the poison had access. At the same time, the muscles were everywhere active, and freely contracted when directly stimulated. The sciatic nerve was then exposed in the tied limb, *above the points of ligature*, and on stimulating it, energetic movements occurred below the knee of that limb, and there only. The heart was, at this time, acting at the rate of fifty per minute.

This experiment was repeated with one grain of iodide of methyl-strychnium, and the same general results were obtained. The evidence that was thus acquired in favour of an action on the peripheral terminations of the motor nerves was strengthened by a modification of this method of experiment.

Experiment XXIX—The right gastrocnemius muscle of a frog was carefully dissected from its connections, excepting that its origin and insertion, and the nerve-fibres entering it, were untouched, and that all its blood-vessels were ligatured. One-tenth of a grain of sulphate of methyl-strychnium, dissolved in five minims of distilled water, was then injected under the skin of the back. Twenty minutes afterwards, the animal being in a perfectly relaxed and motionless condition, the two sciatic nerves were exposed. Galvanism of the left produced no movement in the left limb, while galvanism of the right produced energetic movements of the right limb, which were seen to be due solely to contractions of the right gastrocnemius muscle, the other muscle remaining motionless. At the same time, direct stimulation by galvanism caused contractions as freely in the poisoned muscles as in the non-poisoned right gastrocnemius.

In an experiment, in which iodide of methyl-strychnium was substituted for sulphate, the effects were the same. We have, therefore, demonstrated that the methyl-strychnium derivatives produce paralysis and death by destroying the function of the motor nerve end-organs, and that their mode of action is, therefore, identical with that of curare. This conclusion is an extremely curious and interesting one. It is difficult to imagine a more decided modification in the action of any substance than has been produced by the addition of iodide or sulphate of methyl to strychnia. The striking characteristic of strychnia-action is the great and uncontrollable activity of the muscular system; that of curare, of iodide, and sulphate of methyl-strychnium, and, as we shall presently see, of several other similarly modified poisons, is the flaccid and motionless condition caused by the impossibility of exciting muscular action through the nervous system. So opposite are their effects that physiologists look upon curare as a powerful counteragent to strychnia, while physicians have employed it with success in the treatment of strychnia-poisoning and of tetanus. It is remarkable that by so simple a chemical process so thorough a change should be produced in physiological action.

The experiments we have already described have also shown that this change in chemical constitution has greatly reduced the poisonous activity of strychnia. This effect is still more clearly exhibited in the following table:—

No.*	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.
VII.	Iodide of methyl-strychnium.	Rabbit, 3 lbs. 8 oz.	Subcutaneously.	15 grs. (containing 10·5 grs. of strychnia).	Paralysis in 50 minutes, continuing for more than 2 hours, and followed by recovery.
XIII.	Strychnia (suspended in distilled water).	Do. (same rabbit as in Expt. VII.)	Subcutaneously.	0·05 gr.	Tetanus in 15 minutes; death in 30 minutes.
XVII.	Iodide of methyl-strychnium.	Do., 3 lbs. 13 oz.	By stomach.	30 grs. (containing 21·1 grs. of strychnia).	No effect.
XIX.	Strychnia (as hydrochlorate).	Do., (same rabbit as in Ex. XVII.)	By stomach.	0·1 gr.	Tetanus in 22 minutes; death in 31 minutes.
XXIII.	Sulphate of methyl-strychnium.	Do., 3 lbs. 3½ oz.	Subcutaneously.	0·8 gr. (containing 0.67 gr. of strychnia).	Paralysis in 29 minutes, continuing for 53 minutes, and followed by recovery.
XXXIII.	Sulphate of methyl-strychnium.	Do., 3 lbs. 5¾ oz.	By stomach.	20 grs. (containing 16·8 grs. of strychnia).	No effect.

We have made experiments with nitrate of methyl-strychnium and hydrochlorate of ethyl-strychnium, and have found that their action is identical with that of the iodide or sulphate of methyl-strychnium.

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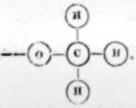
[*] The numbers in this, and in the other short tables that are appended to the description of the physiological action of the derivatives of each alkaloid, have reference, in common with the numbers in the text, to the arrangement in the complete table at the end of the paper.

BRUCIA.

Brucia is a poisonous alkaloid derived from some plants belonging to the genus *Strychnos*. It possesses a physiological action exactly similar in character to that of strychnia, but less in degree.

Iodide of methyl-brucium ($C_{23}H_{26}N_2O_4CH_3I+8H_2O$).—Brucia ($C_{23}H_{26}N_2O_4+4H_2O$) is, like strychnia, a mono-acid nitrile base: here also the structure of the group $(C_{23}H_{26}NO_4)'''$ is unknown, but the action of nitric acid on brucia renders it probable that it contains the radical— $-O-\overset{H}{\underset{H}{C}}-H$. The ethyl-brucium compounds were discovered and described by GUNNING,* and the methyl-brucium compounds by STAHLSCHMIDT.† We prepared the iodide of methyl-brucium by adding excess of iodide of methyl to a saturated solution of brucia in rectified spirit, allowing the mixture to stand for some hours, evaporating, and recrystallising from hot water.

It forms thin white scales, and dissolves in 79 parts of water at 37°C, and in 225 parts of water at 9°C. Its taste resembles that of the corresponding strychnia compound.

When administered by subcutaneous injection, iodide of methyl-brucium was reduced to the form of a very fine powder, and suspended and dissolved in warm distilled water. In a series of experiments, it was found that as much as twelve grains could be thus given to a rabbit without any effect, that fifteen grains produced marked symptoms, and that eighteen grains was about the minimum fatal dose. Its method of action is well shown in the following experiment.

EXPERIMENT XL.—We injected seven and a-half grains of iodide of methyl-brucium, suspended and dissolved in warm distilled water, into each of two cavities (fifteen grains in all) previously formed in the subcutaneous cellular tissue over the abdomen of a rabbit, weighing four pounds. This did not produce the slightest effect until two hours and forty-three minutes after the administration, when the rabbit's movements became sluggish. Shortly after, a difficulty was observed in standing, and this posture soon become impossible on account of the increasing feebleness of the limbs. In three hours and three minutes, the rabbit subsided on the abdomen and chest, with the lower jaw resting on the table. The condition was one of perfect quietness, there being no twitches; and, though frequently tested, the reflex excitability appeared normal.

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* Journal für praktische Chemie, vol. lxvii. p. 46.

† POGGENDORFF'S Annalen, vol. cviii. p. 535 (1859).

It remained on the side when so placed, but unsuccessful resistance was made to this change of position. In three hours and thirty-eight minutes, the flaccid state was even more marked, the position was changed without any resistance on the part of the rabbit, severe pinching only occasionally excited a reflex movement, but the respiratory movements were at the rate of sixty-eight per minute. These symptoms continued for other twenty minutes, when some voluntary movements were made, and soon after, the flaccid condition had nearly disappeared. On the following morning, the animal appeared to be perfectly well.

In the experiment we next give, a fatal dose was administered.

Experiment XLI.—We injected, in all, eighteen grains of iodide of methylbrucium, suspended and dissolved in warm distilled water, into two subcutaneous cavities formed over the abdomen of a rabbit, weighing three pounds and twelve ounces. No result was observed until twenty-seven minutes, when uneasiness was manifested by restless movements, and slight quivers were seen in the muscles of the neck. In thirty minutes, there was great difficulty in supporting the head, which shook tremulously, and frequently fell on the table, where it eventually remained at thirty-two minutes. The body was still supported on the limbs, though by no means steadily. In thirty-seven minutes, it lay on the table and remained on the side, unresisting and flaccid. The respirations were, at this time, at the rate of forty-eight per minute, and were occasionally interrupted by faint quivering movements, but these had no spasmodic character. In forty-five minutes, the respirations were thirty-six per minute, and the heart's contractions 160 per minute. In one hour, the respirations were twenty-five per minute; and irritation of the conjunctiva did not now cause any movements of the eyelids. In one hour and seven minutes, the respiratory movements were irregular and shallow, only about sixteen occurring in the minute, while the heart was contracting at the rate of 120 per minute. The limbs were perfectly flaccid and motionless. The respiratory movements gradually became less apparent, a series of feeble quivers occurred in the muscles of the face, and death immediately afterwards occurred, one hour and thirteen minutes after the administration.

In the autopsy, the cardiac action was found to be regular and rhythmical, though only at the rate of seventy-four per minute. In three minutes after death, galvanism of the sciatic, phrenic, and other nerves, did not produce any muscular contraction; while it was found by direct galvanism that the muscles retained their contractibility for many minutes afterwards. *Rigor mortis* did not occur until more than one hour after death.

For the purpose of contrasting these symptoms with those that are caused by brucia itself, we shall describe, very briefly, an experiment in which the rabbit, that recovered after the administration of fifteen grains of iodide of methylbrucium, was rapidly killed by a somewhat large dose of brucia.

EXPERIMENT XLIII.—One-fifth of a grain of brucia was dissolved in ten minims of very dilute hydrochloric acid, and injected, with WOOD'S syringe, into the subcutaneous tissue of the rabbit that had, some days previously, been subjected to an experiment with fifteen grains of iodide of methyl-brucium. In seven minutes, a constrained position was assumed by the rabbit, and the slightest touch caused a sudden spasmodic contraction of the four limbs by which the body was swiftly elevated. In eight minutes, the rabbit sprang to a considerable height, and fell in a well-marked tetanic convulsion, which lasted about fifteen seconds. After this, a series of violent tetanic convulsions, of a distinctly opisthotonic character, followed each other in rapid succession; and at the termination of one of these, eighteen minutes and thirty seconds after the injection of the poison, the rabbit died. There was distinct *rigor mortis* thirty minutes after death.

For internal administration, the iodide of methyl-brucium was also reduced to a very fine powder, and suspended and dissolved in warm distilled water. It was then introduced into the stomach, by means of a gum-elastic catheter. In this way, we performed several experiments, but never succeeded in producing any effect, although as large a dose as thirty grains was at one time administered. It is well known that there is considerable difficulty in affecting a rabbit by a poison introduced into the stomach. That this difficulty was not due, in the present instance, to any recognised cause peculiar to the stomach of the rabbit, was shown by an experiment in which we produced tetanic symptoms and death by introducing two grains of brucia into the stomach of the rabbit that had previously received thirty grains of iodide of methyl-brucium without any effect whatever.

Sulphate of methyl-brucium ($((C_{23}H_{26}N_2O_4CH_3)_2SO_4$, dried at 100°C) was prepared by precipitating a hot solution of the iodide by means of sulphate of silver. It forms a white crystalline mass, readily soluble in water, and, as well as the iodide, gives the ordinary brucia reaction with nitric acid. It is freely soluble in cold water.

We examined the effects of this substance by subcutaneous injection and by introduction into the stomach. For the former purpose, it was dissolved in a few minims of distilled water, and injected under the skin with a WOOD'S syringe. In a rabbit, one grain could be thus given without any effect, two grains caused marked effects, which were not, however, fatal; while two grains and a-half soon killed the animal. The symptoms were the same as those of the iodide, and, therefore, very different from the exaggerated reflex action, convulsions, and tetanus, which are caused by brucia itself. They are illustrated in the following experiments.

EXPERIMENT LIII.—We injected two grains of sulphate of methyl-brucium, dissolved in fifteen minims of distilled water, under the skin of a rabbit, weighing two pounds and thirteen ounces and three-quarters. In ten minutes, the animal had obviously some difficulty in moving about, and it could not stand steadily. The limbs soon after yielded, and it lay down on the abdomen, chest, and lower jaw; while occasional quivering movements occurred in the muscles of the body. In thirty-four minutes, it lay unresisting and quiet on the side, and the respirations were at the rate of seventy-four per minute. In forty minutes, the respirations were at the rate of fifty-four per minute. It lay in a perfectly relaxed and quiet condition, and when the skin was severely irritated, only extremely feeble reflex movements followed. In one hour and two minutes, the respirations were at the rate of forty-eight per minute; and though irritation of the cornea or conjunctiva did not cause any movement of the eyelids, reflex movements could be excited by severe pinching of the skin. This condition of helpless prostration continued for about thirty minutes, during which some faint twitches of the body and jerking movements of the limbs occasionally occurred. Soon after this, however, a marked improvement was observed: the respirations became fuller and more frequent; irritation of the eyeball was followed by contractions of the eyelids; and, at last, well-directed efforts were made to recover a normal position, and these ultimately proved successful at about two hours after the poison had been injected. The rabbit recovered perfectly.

Experiment LIV.—Two and a-half-grains of sulphate of methyl-brucium was dissolved in fifteen minims of distilled water, and administered by subcutaneous injection to a rabbit, weighing three pounds and fourteen ounces and a-half. In twenty-two minutes, the animal was lying on the abdomen and chest, but the head was still supported by the muscles of the neck; there was distinct congestion of the ears and conjunctiva. In thirty-five minutes, the head had fallen on the table, and the rabbit was perfectly flaccid, and apparently unable to make any voluntary movements. The respirations were at the rate of eighty-two per minute. In fifty-three minutes, the number of the respirations had diminished to twenty-four per minute, while their character was extremely feeble and shallow. In one hour and two minutes, the respiratory movements occurred at long intervals, and were accompanied with a faint tremor of the body and limbs; and it was ascertained that the cardiac contractions were occurring regularly, at the rate of 160 beats per minute. In one hour and ten minutes, the respirations altogether ceased, and death occurred. During the progress of the symptoms, the reflex excitability was frequently tested, with the result that not the slightest increase was ever observed.

The autopsy was immediately made: the heart was found contracting at the rate of 120 per minute; the vermicular action of the intestines was well marked; the conductivity of the sciatic nerves was lost three minutes after death; and idiomuscular irritability persisted for more than twenty minutes afterwards. *Rigor mortis* had not commenced forty minutes after death.

For administration by the stomach, we dissolved this substance in warm distilled water, and introduced the solution through a gum-elastic catheter. We found that as much as twenty grains could be thus given without any effect, and it was not considered advisable to increase this dose. Its magnitude is apparent when we recollect that it contains about seventeen grains of brucia; and we have already seen that when two grains of this alkaloid is introduced into the stomach of a rabbit, the most violent tetanic convulsions are quickly produced, and death soon follows.

The short account we have given of a few of our experiments with iodide and sulphate of methyl-brucium is sufficient to show that these substances have an action that is very different from that of brucia itself. Brucia is a violent convulsant poison, and it causes death by either exhaustion or asphyxia; its methyl derivatives never produce convulsions, nor do they even increase the reflex activity; and although they cause death by asphyxia, this asphyxia, in place of being the result of prolonged and continuous muscular action, due to abnormal nerve activity, is the result of muscular paralysis, due to partial or complete absence of normal nerve activity. We have demonstrated the latter effect by the following experiments, which further show that the influence of the methyl derivatives of brucia is exercised on the terminations of the motor nerves.

EXPERIMENT LVI.—The left iliac artery of a frog, weighing 608 grains, was tied, after exposing it by removing a portion of the sacrum, and one-fifth of a grain of sulphate of methyl-brucium, dissolved in ten minims of distilled water, was then injected into the abdomen. In four minutes, every portion of the frog except the left leg was paralysed. In five minutes and thirty seconds, weak interrupted galvanism, applied to any portion of the skin, caused violent movements of the left leg, and of it alone, every other part of the body remaining motionless. The heart, as ascertained by its impulse, was contracting thirty times per minute. In seven minutes, the right sciatic nerve was exposed—the incisions necessary for which excited energetic reflex movements of the left limb—and on galvanising it, strong contractions of the left limb occurred, but no movement occurred in the right limb. The muscles were everywhere in a normal state, and freely responded to direct galvanic stimulation; and the heart still contracted at the rate of thirty beats per minute.

In a similar experiment, with half a grain of iodide of methyl-brucium, the same effects were observed. It is, therefore, apparent that these substances do not directly influence the action of the heart, of the muscles, of the spinal cord, or of the sensory (afferent) nerves, but that the paralysis, which they so prominently cause, is the result of an action on the motor nerves. In the above experiment, the whole course of the sciatic nerve, from the pelvis to the extremity of the left posterior limb, was protected from the influence of the poison. The experiment does not, therefore, show if the methyl-brucium compounds have an elective action for any special portion of the nerve. In the next experiment, a much more limited portion of the nerve was protected from the poisonous action.

EXPERIMENT LVII.—In a frog, weighing 542 grains, the right gastrocnemius muscle was exposed; the muscle was separated from all its connections, excepting its origin and insertion and the nerve-fibres that entered it. One-sixth of a grain of sulphate of methyl-brucium, dissolved in ten minims of distilled water, was then injected into the abdomen. In twenty minutes, a condition of complete paralysis was present everywhere except in the right leg. The two sciatic nerves were exposed, and on galvanising the left nerve, feeble movements occurred in the right leg, and there only. When the right nerve was galvanised, movements occurred in the right leg, which were observed to be solely due to contractions in the right gastrocnemius muscle.

In this experiment, the terminations of the sciatic nerve in the right gastrocnemius muscle were alone protected from the direct influence of sulphate of methyl-brucium. This substance had access to all the other terminations of the right sciatic nerve, to the trunk of this nerve, and to all the other nerves of the body. No manifestation of vitality was obtained anywhere, except in the right limb, and it was restricted to contractions of one muscle of that limb. As these contractions could be produced by a stimulus originated in and conducted along the nerve trunk, it is obvious that the vitality of this portion of the nerve was not lost. And as the stimulus produced no effect on the terminations of the nerves to which sulphate of methyl-brucium had access, while it produced an effect on those that were protected from its direct influence, it is evident that this poison acts on the peripheral terminations of the motor nerves.

The physiological action of brucia is, therefore, completely changed by the addition of iodide or sulphate of methyl. It is also apparent that its activity as a poison is greatly lessened; and the following table, which contains a succinct statement of some of the previously-mentioned facts, will clearly illustrate this:—

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.
XL.	Iodide of methyl-brucium.	Rabbit, 4 lbs.	Subcutaneously.	15 grs. (containing 8·7 grs. of dry brucia).	Paralysis in 3 hours and 3 minutes, continuing for more than 28 minutes, and followed by recovery.
XLIII.	Brucia (crystallised).	Do. (same rabbit as in Experiment XL.)	Subcutaneously.	0·2 gr. (containing 0·17 gr. of dry brucia).	Tetanus in 8 minutes; death in 18 minutes 30 seconds.
XLVI.	Iodide of methyl-brucium.	Do., 4 lbs. 2 oz.	By stomach.	30 grs. (containing 17·4 grs. of dry brucia).	No effect.
LI.	Brucia (crystallised).	Do. (same rabbit as in Experiment XLVI.)	By stomach.	2 grs. (containing 1·7 gr. of dry brucia).	Tetanus in 44 minutes; death after 3 hours.
LIII.	Sulphate of methyl-brucium, dried.	Do., 2 lbs. 13¾ oz.	Subcutaneously.	2 grs. (containing 1·7 gr. of dry brucia).	Paralysis in 20 minutes, continuing for about 1 hour and 40 minutes, and followed by recovery.
LVIII.	Sulphate of methyl-brucium, dried.	Do., 4 lbs. 2 oz.	By stomach.	20 grs. (containing 17·2 grs. of dry brucia).	No effect.

THEBAIA.

One of the active principles of opium possesses an action in all respects the same in character as that of strychnia or brucia. We principally owe our knowledge of the method in which thebaia acts to the admirable researches of CLAUDE BERNARD. This distinguished physiologist has further demonstrated that thebaia does not possess any soporific property, that it is the most active toxic principle in opium, and that it ranks first among the alkaloids of this drug that have a convulsant action.* From our experience of its properties, we should assign to it a lower rank than brucia as a toxic and convulsant substance.

Iodide of methyl-thebaium.—The close analogy in physiological action that exists between thebaia ($C_{19}H_{21}NO_3$) on the one hand, and strychnia and brucia on the other, led us to subject this alkaloid to the action of iodide of methyl. The method adopted was the same as that described for the preparation of iodide of methyl-brucium, and the reaction takes place as readily. The product crystallises from alcohol in hard, shining, transparent crystals, which, when air-dried, have the composition ($C_{19}H_{21}NO_3CH_3I$). They dissolve in 16·5 parts of water at 37°C., and in 63·5 parts of water at 9°C.† When a hot, saturated, aqueous solution is allowed to cool, it gelatinises, and the jelly, when left to itself, in some hours, and, when stirred, in a few minutes, is converted into a mass of minute silky needles, which when dried in the air, have the same composition as the crystals obtained from the alcoholic solution.

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* Comptes Rendus, vol. lix. 1864, p. 413.

† The methyl derivatives of thebaia have not been described. We shall take some other opportunity of giving details of their chemical relations.

It is much more soluble in water than the iodides of methyl-strychnium and methyl-brucium, and, on this account, we commenced its administration in relatively small quantities. We found that doses of one, five, and six grains, nearly completely dissolved in very dilute spirit, and administered to rabbits by subcutaneous injection, produced absolutely no effect. When, however, the dose was increased to ten grains, partial and then complete paralysis was caused, and death quickly occurred; while serious symptoms were caused by eight grains, but they did not terminate fatally. We shall give some details of these two experiments.

Experiment LXII.—We dissolved eight grains of iodide of methyl-thebaium in very dilute alcohol, and injected the solution, with Wood's syringe, into the subcutaneous cellular tissue of a rabbit, weighing two pounds and twelve ounces. Symptoms of uneasiness occurred in thirty minutes, and were soon followed by quivering movements of the head and ears, and, to a slight extent, of the rest of the body. It was soon apparent that the neck muscles were scarcely able to support the head, for it frequently fell on the table,but the rabbit did not permit it to remain there until forty-five minutes after the administration. At this time, the respirations were at the rate of seventy-eight per minute, and, although the head was resting on the table, the body of the animal was supported, in a comparatively normal posture, on the limbs. There were occasional tremulous movements of the body, but no exaggeration of the reflex function could be discovered. The rabbit remained in this state for about thirty minutes; but soon after this, the tremulous movements disappeared, the head was raised and supported normally, and a perfectly natural posture was assumed. Every symptom had disappeared within two hours after the administration.

Experiment LXIII.—Ten grains of iodide of methyl-thebaium, reduced to a very fine powder, was partially dissolved and partially suspended in very dilute alcohol, and injected under the skin of a rabbit, weighing two pounds and eleven ounces. There was no obvious effect until ten minutes, when it was observed that the animal moved with difficulty. Tremulous movements then occurred, the limbs occasionally yielded, and the head frequently fell. In twelve minutes, the rabbit lay on the abdomen and chest, with the lower jaw resting on the table; and the tremulous movements only occurred at intervals. It could now be lifted without any struggles. In nineteen minutes, the condition was one of complete flaccidity, the only movements were an occasional gasping respiration, but common sensibility was still retained. It continued thus, on the very verge of death, for about four minutes, when a few quivering contractions occurred in the muscles of the face and neck, and the respirations altogether ceased. During the course of the symptoms, there was never the slightest trace of any exaggeration in the reflex activity, nor of spasmodic or convulsive movements.

The autopsy was immediately made. The heart was found contracting, in normal rhythm, at the rate of eighty-five per minute, and its spontaneous contractions did not cease until eight minutes; and the intestinal peristalsis was active. The sciatic nerves were exposed four minutes after death, and stimulated with weak and strong currents of interrupted galvanism, but no muscular contractions were thereby caused. The muscles themselves readily contracted when the poles were applied directly to their surface, and continued to do so for more than fifteen minutes after death. There was no appearance of *rigor mortis* one hour and five minutes after death, and the muscles were, at this time, alkaline in reaction.

We administered to the rabbit, which had survived the administration of eight grains of iodide of methyl-thebaium (Experiment LXII.), a fatal dose of the thebaia from some of which the methyl compound had been prepared. The striking contrast in the symptoms that were produced will be seen from the following account of the Experiment.

EXPERIMENT LXVI.—We injected one-fifth of a grain of thebaia, dissolved in very dilute hydrochloric acid, into the subcutaneous cellular tissue of the rabbit, which had been subjected to an experiment, some days previously, with eight grains of iodide of methyl-thebaium. The injection did not appear to cause much annoyance, as the animal jumped about naturally for forty minutes after it. Soon after, however, its movements became more constrained and cautious, and occasional twitches occurred in the muscles of the back. These gradually became more marked and powerful, and in forty-eight minutes, they assumed the character of spasmodic starts. In forty-nine minutes, a touch, even when very gentle, of any portion of the skin excited a violent spasmodic jump, and in fifty-two minutes, a spontaneous violent opisthotonic convulsion took place, and continued for forty-five seconds. The rabbit now lay on its side; every respiratory movement provoked a short fit of tetanus, while, occasionally, a violent and prolonged fit occurred. This condition lasted for two minutes, when, at the termination of one of the more violent of these fits, death occurred,—fifty-four minutes after the administration of the poison.

It was found, in the autopsy, that the sciatic nerves retained their motor conductivity for at least fifteen minutes after death. A certain degree of muscular rigidity was observed at twenty-eight minutes, and *rigor mortis* was perfectly established at forty minutes, when all the muscles were acid in reaction, although the temperature of the abdominal cavity was as high as 95° F.

The internal administration of iodide of methyl-thebaium was effected in the same way as we have described for the corresponding strychnia and brucia compounds. It was found that, with this substance also, so large a dose as thirty grains could be introduced into the stomach of a rabbit without any effect. Well-marked symptoms were produced in the same animal by three, and, on another occasion, by three and a-half grains of thebaia similarly administered, but it recovered after both doses. Four grains was, however, a fatal dose, as will be seen from the following Experiment.

Experiment LXXIV.—Four grains of thebaia, almost completely dissolved in very dilute hydrochloric acid, was introduced, by a gum-elastic catheter, into the stomach of the rabbit that had received thirty grains of iodide of methyl-thebaium (Experiment LXXI.) In six minutes, a violent tetanic convulsion occurred; after this, the rabbit remained on the side, and convulsion succeeded convulsion until its death, nineteen minutes after the administration of thebaia. *Rigor mortis*, with an acid reaction of the muscles, was completely established at thirty-seven minutes after death.

Sulphate of methyl-thebaium ($((C_{19}H_{21}NO_3CH_3)_2SO_4$, dried at 100° C.), was prepared by precipitating an aqueous solution of the iodide by means of sulphate of silver. It forms a white, indistinctly crystalline mass. It dissolves readily in water, and gives, with sulphuric acid, the reaction of thebaia.

We found it to be a less active substance than the corresponding derivative of either strychnia or brucia, as doses of four and of four-and-a-half grains were not fatal, though they produced symptoms, when injected into the subcutaneous cellular tissue of rabbits. Five grains appears to be about the smallest quantity that can produce death when administered to rabbits in this manner. The experiments in which four and a-half and five grains were given are sufficient to illustrate the general physiological effects of this substance.

Experiment LXXVII.—We dissolved four and a-half grains of sulphate of methyl-thebaium in fifteen minims of distilled water, and injected this solution into the subcutaneous tissue at the flank of a rabbit, weighing three pounds and eleven ounces and a-half. In seventeen minutes, the rabbit had some difficulty in jumping about, for it occasionally stumbled, and rested for a few seconds on the chest. In twenty-one minutes, it was lying on the abdomen, with the lower jaw resting on the table; and, occasionally, a series of shivering tremors took place in the muscles of the back. In thirty minutes, it remained on the side, when so placed, and was perfectly flaccid. The respirations were at the rate of sixty per minute. In forty-one minutes, the respirations had diminished in frequency to forty per minute, and during inspiration the abdominal muscles contracted in a tremulous manner. In fifty-five minutes, the respirations had increased in number to seventy-one per minute, and in one hour and thirty minutes, they appeared to have regained their normal rapidity; but it was impossible to ascertain this definitely, on account of frequent interruptions by tremulous movements of the abdominal muscles. The rabbit was still lying on the side in a perfectly flaccid state. In one hour and thirty-two minutes, however, it suddenly raised the head, rose, and assumed a normal posture; but the trembling continued. This trembling, very faint and not at all spasmodic, was the last symptom to disappear, which it did about two hours and thirty minutes after the injection of the poison. We frequently tested the condition of the reflex activity, and did not find it increased at any period during the experiment.

EXPERIMENT LXXVIII.—Five grains of sulphate of methyl-thebaium was dissolved in thirty minims of distilled water, and injected under the skin of a rabbit, weighing four pounds and half an ounce. Its effects began to be seen in thirteen minutes, when, after a few restless movements, the rabbit subsided on the abdomen and chest. Complete flaccidity soon after occurred; and the respirations became shallow and gasping, and they diminished in frequency until, at twenty-five minutes after the injection, they were only at the rate of twenty-three per minute. Occasional, very weak, tremulous movements occurred at this time. In thirty-five minutes, severe pinching of the skin caused only a feeble reflex movement, while the contraction of the eyelids, after irritation of the eyeball, was almost imperceptible. The rabbit appeared still to retain consciousness. In fifty minutes, no movement followed severe pinching of the skin, or irritation of the eyeball, and the respirations were gasping and infrequent. In fifty minutes, a few twitches occurred in the muscles of the face, and either immediately before or during these the rabbit expired.

In the autopsy, which was immediately performed, the heart was seen contracting at the rate of seventy-eight per minute, and the intestinal peristalsis seemed normal. Four and a-half minutes after death, neither a weak nor a powerful galvanic current could excite any muscular contraction when applied to the trunk of a sciatic nerve; but idio-muscular irritability was not lost for many minutes after this. At two hours and thirty minutes after death, the rabbit was still perfectly flaccid, and there was not the slightest appearance of muscular rigidity.

We have not observed any symptoms follow the internal administration of this substance, as no effect was produced when we introduced twenty grains, dissolved in warm water, into the stomach of a rabbit. It has been shown by Experiment LXXIV. that four grains of thebaia is a fatal dose when thus exhibited.

The experiments we have narrated contain the most satisfactory proof that the chemical addition of iodide and sulphate of methyl has produced a complete change in the physiological action of thebaia. The nature of the change appears to be identical with that we have described as occurring under similar circumstances in strychnia and brucia. Thebaia acts in the same way as these alkaloids; for it causes increase of the reflex activity, convulsions, and tetanus by an action on the spinal cord. The action of iodide and sulphate of methyl-thebaium is strikingly different; for they diminish reflex excitability, and produce a condition of paralysis in which death occurs by asphyxia. This paralysis, as we have seen, is dependent on an effect on the spinal nerve system.

We will now describe an experiment in which we endeavoured to determine what portion of this system is affected.

EXPERIMENT LXXIX.—The sciatic artery and the two principal veins were tied in the right thigh of a frog, weighing 420 grains, and one-fifth of a grain of sulphate of methyl-thebaium, dissolved in seven minims of distilled water, was injected into the abdominal cavity. In six minutes, the animal was flaccid and motionless, and in other four minutes the respiratory movements of the chest and abdomen had ceased, while those of the throat continued, and did so for several minutes longer. In sixteen minutes, galvanic stimulation by an interrupted current, applied to any portion of the skin, caused movements of the right leg *below the points of ligature*, but nowhere else. In twenty-one minutes, the left sciatic nerve was exposed, and on galvanising it, energetic movements occurred in the right leg, while the left leg and every other part of the body remained motionless. The heart was now contracting at the rate of thirty-six beats in the minute. The muscles that had been laid bare in the left leg, by the dissection necessary for the exposure of the left sciatic nerve, were stimulated by the direct application of an interrupted galvanic current, and they contracted powerfully. This condition continued during other two days; on the second day, even a feeble stimulus applied to the left sciatic nerve was followed by well-marked contractions of the right leg, below the points of ligature; while it caused no movements in those parts of the frog that had been directly acted upon by the poison, although the muscles everywhere contracted when directly stimulated.

We learn from this experiment that sulphate of methyl-thebaium produces paralysis by destroying the conductivity of the motor nerves, and not by interfering with the function of the spinal cord, or of the sensory (afferent) nerves. The next experiment was made with the view to determine what portion of the motor nerve is paralysed by this substance.

EXPERIMENT LXXX.—The left gastrocnemius muscle was exposed in the leg of a frog, weighing 604 grains. The blood-vessels that entered it were ligatured or twisted, and it was carefully separated from all its connections, excepting that its origin and insertion were untouched, and that the nerve fibres that entered it were not divided. Immediately after this somewhat tedious preparation, one-fifth of a grain of sulphate of methyl-thebaium, dissolved in ten minims of distilled water, was injected in the abdomen. Omitting the details of the effects that ensued, it is sufficient to mention that, at thirty minutes after this injection, the sciatic nerve was exposed in each thigh and galvanised, with the result that in the case of the right nerve movements followed in the left leg alone, and in the case of the left nerve movements followed in the left leg, and there only. It was seen that these movements in the left leg were entirely caused by contractions of the left gastrocnemius muscle, that is, of the muscle which had been protected from the direct influence of the poison.

We obtained the same results on repeating these experiments with half-grain doses of iodide of methyl-thebaium.

These experiments demonstrate clearly that the methyl derivatives of thebaia produce their principal physiological effects by impairing and destroying the function of the peripheral termination of the motor nerves—an action that is very different from that of thebaia itself. They also differ from thebaia in being considerably less potent as poisons. Several of these characters are summarised in the appended table.

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition	Dose.	Effect.
LXII.	Iodide of methyl-thebaium.	Rabbit, 2 lbs. 12 oz.	Subcutaneously.	8 grs. (containing 5·5 grs. of thebaia).	Paralysis in 45 minutes, continuing for about 30 minutes, and followed by recovery.
LXVI.	Thebaia.	Do. (same rabbit as in Experiment LXII.)	Subcutaneously.	0·2 gr.	Tetanus in 52 minutes, and death in 54 minutes.
LXXI.	Iodide of methyl-thebaium.	Do., 4 lbs. 6 oz.	By stomach.	30 grs. (containing 20·6 grs. of thebaia).	No effect.
LXXIV.	Thebaia.	Do. (same rabbit as in Exp. LXXI.)	By stomach.	4 grs.	Tetanus in 6 minutes, and death in 19 minutes.
LXXVII.	Sulphate of methyl-thebaium.	Do., 3 lbs. 11½ oz.	Subcutaneously.	4·5 grs. (containing 3·7 grs. of thebaia).	Paralysis in 21 minutes, continuing for 2 hours and 9 minutes, and followed by recovery.
LXXXI.	Sulphate of methyl-thebaium.	Do., 4 lbs. 4 oz.	By stomach.	20 grs. (containing 16·6 grs. of thebaia).	No effect.

CODEIA ($C_{18}H_{21}NO_3 + H_2O$).

We have examined the effect of the addition of iodide and sulphate of methyl to codeia—an opium alkaloid, which, according to CLAUDE BERNARD, is the second in toxic activity, and possesses distinct convulsant but feeble soporific properties.*

Iodide of methyl-codeium.—How† obtained by the action of iodide of ethyl on codeia, iodide of ethyl-codeium, and from it a number of ethyl-codeium compounds, and proved that codeia is a nitrile base. As was to be expected, iodide of methyl acts even more readily on codeia.‡ It is only necessary to heat codeia with a little alcohol and an excess of iodide of methyl to 100° C. for an hour, in a sealed tube, to complete the reaction. The excess of iodide of methyl is distilled off, the alcohol evaporated, and the product crystallised from hot water. It forms large transparent prisms, soluble in 14·5 parts of water at 37° C., and in 49 parts of water at 9° C. Its solution is not precipitated by caustic potash, and in all respects, except in the appearance of its crystals, agrees with iodide of ethyl-codeium.

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* Comptes Rendus, vol. lix. (1864) p. 413.

† Chemical Society's Quarterly Journal, vol. vi. (1853) p. 134.

‡ We shall give details of the chemical relations of the methyl derivatives of codeia on some other occasion.

As iodide of methyl-codeium is tolerably soluble in warm water, we could administer it by subcutaneous injection in the form of solution. It was found, in rabbits, that a dose of five grains was quite inert, that one of fifteen grains caused prolonged and serious symptoms which were recovered from, and that one of twenty grains produced death in a short time. The following details include the principal symptoms that appeared when fifteen and twenty grains were thus administered.

Experiment LXXXIV.—Fifteen grains of iodide of methyl-codeium was dissolved in some warm distilled water, to which a few drops of rectified spirit had been added, and the solution was injected into the subcutaneous cellular tissue of a rabbit, weighing two pounds and fourteen ounces. The rabbit remained sitting quietly until twenty-two minutes afterwards, but in a few seconds more it had some difficulty in retaining a sitting posture, and, on standing, the fore-limbs occasionally yielded, until, at twenty-five minutes, it subsided on the abdomen, chest, and lower jaw. In thirty minutes, it remained on the side without struggling; and now, after considerable intervals, faint twitches occurred in the body and limbs, which, however, had no convulsive character. In thirty-seven minutes, irritation of the cornea or conjunctiva did not cause any movement in the eyelids, but the respirations, though weak, shallow, and somewhat jerking, were at the rate of sixty-seven in the minute. In forty-five minutes, the frequency of the respirations had diminished to sixty in the minute, and there were now no twitches. The rabbit continued to lie in this flaccid state for about two hours longer; at the end of which time, twitches reappeared, at first extremely faint, but, by-and-by, of considerable strength, and involving the muscles of the abdomen, chest, neck, and limbs. In four hours and twenty minutes, the rabbit was again in a perfectly quiet state, the twitches had disappeared, and the common sensibility was in a normally active condition. Frequent attempts were made, soon after, to recover a natural position, and success was at length attained, four hours and twenty-five minutes after the injection of the poison. There were no further symptoms.

Experiment LXXXV.—We injected twenty grains of iodide of methyl-codeium, dissolved as in the preceding experiment, into the subcutaneous cellular tissue of a rabbit, weighing two pounds and twelve ounces and a-half. The animal began to tremble in thirteen minutes, and the head, after being unsteadily supported for a short time, fell on the table. In fifteen minutes, the rabbit remained on the side; the respirations were weak and irregular, and slight starts occurred occasionally. Severe irritation of the skin was now required to cause even an extremely feeble reflex movement. In twenty-four minutes, no movement followed irritation of either the skin or eyeball, and the respirations were mere gasping jerks. In thirty-two minutes, a series of feeble twitches occurred in the face-muscles, and then the respirations entirely ceased.

We immediately exposed the sciatic nerves, and examined their condition: when they were stimulated with galvanism, slight movements followed in the hind limbs at one minute after death; but no movement could be excited at one minute and thirty seconds. The heart was found to be contracting in regular rhythm, at the rate of eighty-two in the minute. Forty-five minutes after death, the body was perfectly flaccid, and there was not the slightest appearance of muscular rigidity.

We may best display the marked differences between these physiological effects and those that are caused by codeia, by describing an experiment in which the rabbit that survived the administration of fifteen grains of iodide of methyl-codeium, was quickly killed by the subcutaneous injection of one grain of codeia.

EXPERIMENT LXXXIX.—We dissolved one grain of codeia in some warm distilled water, to which a few drops of rectified spirit had been added, and injected the solution into the subcutaneous tissue of the rabbit, which was some days previously the subject of Experiment LXXXIV. In fifteen minutes, faint twitches occurred in some of the muscles of the back; and, soon after, a slight touch excited a violent start. Spontaneous spasmodic starts now followed each other, until one hour and eleven minutes, when a violent tetanic convulsion of an opisthotonic character occurred. For some time before this, it was observed that the hind limbs trailed slightly when movements were attempted, indicating, apparently, a slight degree of motor paralysis. The first tetanic convulsion was followed by trismus, which lasted for a few seconds, and by a succession of slight spasms; and soon after its occurrence, unsuccessful efforts were made to recover a normal position. In one hour and thirty minutes, a second violent tetanic convulsion took place, and this presented the character of emprosthotonos rather than of opisthotonos. Such convulsions now recurred after intervals of a few minutes, and at the termination of one of them, one hour and forty-five minutes after the administration of the poison, the rabbit died. In fifteen minutes after death, strong *rigor mortis* was present.

We introduced iodide of methyl-codeium into the stomach of rabbits on two occasions. In one of these, fifteen grains were thus administered, and in the other, thirty grains; but no effect was produced by either dose. Codeia itself, however, is by no means a violent poison when given to rabbits in this manner. We made a considerable number of experiments, but did not succeed in causing death even with fifteen grains. In the following experiment we employed ten grains.

EXPERIMENT XCIV.—By means of a gum-elastic catheter, we injected ten grains of codeia, dissolved in warm distilled water to which a few drops of dilute hydrochloric acid had been added, into the stomach of a rabbit, weighing three pounds and thirteen ounces. In twenty-four minutes, some symptoms of sleepiness were observed, which chiefly manifested themselves by nodding movements of the head. In thirty-nine minutes, the reflex excitability seemed increased, as a slight touch caused a sudden, somewhat spasmodic start. In one hour, the sleepy condition had so far increased, that the head rested on the table, and the eyelids were semi-closed. In one hour and twenty minutes, the rabbit could be placed in almost any position, provided physical rest were allowed; and it would remain sleeping in these attitudes until roused by sounds or by pretty violent irritations. It continued in this condition for more than two hours; but in three hours, the sleepiness was less marked, and on the following morning the rabbit was in a perfectly natural state.

Sulphate of methyl-codeium was prepared from the iodide, by precipitating it by means of sulphate of silver. It forms a white crystalline mass, readily soluble in cold water.

It is a rather more active poison than the iodide, for we found that ten grains, exhibited subcutaneously, was sufficient to kill a rabbit. We observed only slight symptoms with eight grains.

EXPERIMENT XCVI.—Eight grains of sulphate of methyl-codeium was dissolved in twenty minims of distilled water, and injected under the skin of a rabbit, weighing four pounds. No distinct effect was observed until thirty minutes, when some uneasiness was shown by restless movements of the limbs, and, soon after, a little trembling occurred. Weakness of the limbs was then exhibited by occasional stumbles, and, in thirty-three minutes, the rabbit fell, and remained resting on the abdomen, with the lower jaw on the table. There were no starts nor spasms, and even the trembling had now ceased; while severe irritation of the skin caused merely slight reflex movements. After remaining in this state for twenty minutes, the symptoms gradually improved, and the rabbit appeared to be quite well two hours after it had received the poison.

EXPERIMENT XCVII.—Ten grains of the sulphate of methyl-codeium was dissolved in distilled water, and injected under the skin of a rabbit, weighing four pounds and four ounces. In twenty-three minutes, the head and portions of the body shook in a quivering manner; and, gradually, the head sank until it rested on the table. In twenty-five minutes, the legs gave way, and the animal fell; faint twitches occurred over the body, but otherwise the condition was one of complete flaccidity. In thirty-five minutes, it remained on the side, without any resistance. In thirty-eight minutes, the respirations were laboured, and at the rate of thirty-six per minute; and in other four minutes, they had fallen to twenty per minute. In forty-one minutes, these movements were extremely shallow and irregular; and in forty-two minutes, they altogether ceased. In the course of this experiment, no convulsive symptoms occurred, and no hypnotism was observed.

In the autopsy, the heart was seen acting, one minute after death, at the rate of 160 beats per minute, and the intestinal peristalsis was found to be normal. The motor conductivity of the sciatic nerves was retained at three minutes after death, but it had disappeared in other four minutes; while the idio-muscular irritability was not lost until more than sixty minutes after death.

For internal administration, we followed the plan already described. No symptom whatever was observed when the large dose of twenty grains was introduced into the stomach of a rabbit. We did not, accordingly, consider it advisable to continue this method of administration any further.

As we have already stated, and as the experiments we have narrated clearly show, the principal effects that are caused by codeia are convulsions and hypnotism. In our experiments with rabbits, the latter effect was manifested only when large doses were introduced into the stomach. It was not seen when this alkaloid was administered by subcutaneous injection, probably because sleep was then prevented by the spasmodic starts and convulsions that were so prominently caused. We learn from our experiments that the iodide and sulphate of methyl-codeium have a very different action from codeia. We have never observed any hypnotic effect follow their administration, and, in place of convulsions, we have seen that they produce paralysis. This, indeed, is the only marked symptom that follows their administration, and it is apparent that it does not depend on an effect on the muscles, nor on the cerebral lobes. We endeavoured to determine the exact cause of this paralysis by experiments with localised poisoning on frogs.

EXPERIMENT XCVIII.—Having tied the right sciatic artery and vein of a frog, weighing 722 grains, one grain of sulphate of methyl-codeium, dissolved in distilled water, was injected into the abdominal cavity. In fifteen minutes, voluntary movements had disappeared, and the frog was lying on the abdomen, in a flaccid state. In thirty minutes, pinching of the skin with a pair of forceps excited movements in all the limbs, but these were most energetic in the right posterior extremity. In one hour and thirty minutes, similar stimulation excited no movement except in the right posterior extremity (where the vessels had been tied). The application of an interrupted galvanic current to the exposed trunk of the left sciatic nerve was now followed by active movements of the right leg, but of no other part; while, at the same time, the muscles in the poisoned regions freely responded to galvanic stimulation directly applied to them. In two hours and forty minutes, the condition was the same, and, judging from the cardiac impulse, the heart was contracting at the rate of thirty-five per minute.

We need not again enter into the reasons for concluding from such an experiment that the paralysis caused by sulphate of methyl-codeium is due to an action on the motor nerves. As has been already done with the corresponding substances treated of in the previous portion of this paper, we, in the next place, determined what portion of the motor nerve—trunk or periphery—is acted on.

Experiment XCIX.—In a frog, weighing 694 grains, the left gastrocnemius muscle was prepared in the manner described in Experiments XXIX., LVII., and LXXX., and one grain of sulphate of methyl-codeium, dissolved in distilled water, was injected into the abdomen by means of a Wood's syringe. In one hour after this, a condition of flaccidity being present, the trunks of the two sciatic nerves were exposed, and stimulated with an interrupted galvanic current. When the right nerve was thus treated, some contractions followed in the left leg, and nowhere else; and when the left nerve was thus treated, vigorous contractions followed in the left leg; and it was observed that the movements of the left leg were caused by contractions restricted to its gastrocnemius muscle, that is, the muscle to which the poison had no direct access. At this time, the muscles in all parts of the body contracted freely when the poles of the battery were applied to their surfaces, and continued to do so for many hours longer.

We repeated these last experiments with iodide of methyl-codeium, and obtained the same general results.

We have, therefore, demonstrated that iodide and sulphate of methyl-codeium produce paralysis, by destroying the function of the peripheral terminations (end-organs) of the motor nerves—a mode of action that distinguishes them, as physiological agents, in a most striking manner from codeia. It will also be seen from the following table, that the poisonous (toxic) activity of the codeia in these methyl-compounds is considerably diminished.

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition	Dose.	Effect.
LXXXIV.	Iodide of methyl-codeium.	Rabbit, 2 lbs. 14 oz.	Subcutaneously.	15 grs. (containing 10·2 grs. of dry codeia).	Paralysis in 25 minutes. continuing for about 3 hours, and followed by recovery.
LXXXIX.	Codeia (crystallised).	Do. (same rabbit as in Ex. LXXXIV.)	Subcutaneously.	1 gr. (containing 0·94 gr. of dry codeia)	Spontaneous twitches in 15 minutes, tetanus in 1 hour and 11 minutes, and death in 1 hour and 45 minutes.
XCI.	Iodide of methyl-codeium.	Do., 2 lbs. 13 oz.	By stomach.	30 grs. (containing 20·3 grs. of dry codeia).	No effect.
XCIV.	Codeia (crystallised).	Do. (same rabbit as in Ex. XCI.)	By stomach.	10 grs. (containing 9·4 grs. of dry codeia).	Sleepiness in 24 minutes, increase of reflex excitability in 39 minutes, and followed by recovery in more than 3 hours.
XCVI.	Sulphate of methyl-codeium.	Do., 4 lbs.	Subcutaneously.	8 grs. (containing 6·6 grs. of dry codeia).	Paralysis in 33 minutes, continuing for more than 20 minutes, and followed by recovery.
C.	Sulphate of methyl-codeium.	Do., 4 lbs. 13 oz.	By stomach.	20 grs. (containing 16·5 grs. of dry codeia).	No effect.

MORPHIA.

The most recent and trustworthy investigations show that, among the opium alkaloids, morphia ($C_{17}H_{19}NO_3 + H_2O$) is next in activity as a soporific to narceia, that it possesses a less convulsant action than codeia, and that its fatal dose is one of the largest of those of the active principles of opium.*

Iodide of methyl-morphium ($C_{17}H_{19}NO_3CH_3I$).—How subjected morphia to the action of iodide of ethyl and of iodide of methyl, prepared and described a number of the ethyl-morphium and methyl-morphium compounds, and proved that morphia is a nitrile base.† We prepared the iodide of methyl-morphium by How's method, viz., by treating morphia with alcohol and an excess of iodide of methyl in a sealed tube, at 100° C., for an hour, distilling off the excess of iodide of methyl, and recrystallising from hot water.

It forms long, transparent, prismatic needles; and dissolves in 34 parts of water at 37° C., and in 88·5 parts of water at 9° C.

As it is well known that comparatively large doses of morphia are required to produce any symptom in such animals as rabbits, we at once commenced the administration of iodide of methyl-morphium in very large doses. We were unable to produce any effect whatever when so large a dose as twenty grains was injected under the skin of a small rabbit; and, as this could only be administered as a fine powder, suspended in warm distilled water, it was extremely inconvenient to give any larger quantity in a form necessarily so bulky. Eight grains of morphia was afterwards exhibited, in the same way, to this rabbit, and it caused the usual symptoms and death. It may be interesting and satisfactory to give some details of these two experiments.

EXPERIMENT CI.—Twenty grains of iodide of methyl-morphium was reduced to a fine powder, mixed with two drachms of warm distilled water, and injected into two previously formed subcutaneous cavities at the flanks of a rabbit, weighing two pounds and fourteen ounces. The rabbit was carefully observed for four hours, but no symptom occurred during this time. It was perfectly well on the following morning.

EXPERIMENT CV.—Eight grains of morphia, suspended in warm distilled water, was introduced into the subcutaneous cellular tissue of the rabbit that had been employed, two days previously, in Experiment CI. In one hour and four minutes, an inclination to sleep was observed, the eyelids closed, and the head sank on the table, but a slight sound immediately roused the rabbit. In two hours, the soporific effect was more marked; and the animal remained in almost any position in which it could be placed, provided the change was made gradually and gently;

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* CLAUDE BERNARD, Comptes Rendus, vol. lix. 1864, p. 413.

† Chemical Society's Quarterly Journal, vol. vi. (1853) p. 126.

and, however unnatural the position might seem to be, if it were consistent with rest, sleep immediately occurred. In three hours, there was some difficulty in rousing it, and when this was done, it remained awake for a few seconds only. In six hours, the respirations had fallen to the slow rate of twenty-six in the minute. This condition lasted, altogether, for about forty-eight hours, when spasms made their appearance, which, by-and-by, assumed all the characters of epileptiform convulsions. These epileptic fits frequently recurred, and could be excited, at any time, by pinching the skin. They consisted of tonic spasms of the limbs and of the abdominal muscles, followed by twisting of the head to the right, grinding movements of the lower jaw, and violent opisthotonos. The rabbit was found dead on the morning of the third day after the administration.

The two subcutaneous cavities into which the morphia had been introduced were laid open, and a small quantity of unabsorbed morphia was found in both. The cavities into which iodide of methyl-morphium had been introduced were also laid open, but none of this substance was found.

We were unsuccessful in producing any symptoms by the internal administration of iodide of methyl-morphium. Thirty grains was found to be perfectly inert when exhibited by the stomach, while the same rabbit was decidedly narcotised with five grains of morphia similarly exhibited. It is interesting, for the purpose of comparison, to give a short account of these two Experiments.

Experiment CVI.—We suspended thirty grains of finely-powdered iodide of methyl-morphium in distilled water, and injected the mixture into the stomach of a rabbit, weighing three pounds and twelve ounces. It was observed for more than two hours, but no symptoms could be detected.

Experiment CVII.—We suspended five grains of finely-powdered morphia in distilled water, and injected the mixture into the stomach of the rabbit that was used, two days previously, in Experiment CVI. In one hour and six minutes, the rabbit was observed to be sleepy, and it soon after laid its head on the table. This sleepy condition became gradually more marked: in one hour and twenty-five minutes, the rabbit could be placed in almost any position, and slept thus; while about the same time, a condition resembling that of catalepsy was present, for when we placed the rabbit on the back and raised the fore legs perpendicularly upwards, it remained in this extraordinary attitude for several minutes. In two hours and forty-one minutes, it was observed that the pupils, which were small, did not contract on the approach of a bright light, nor did this stimulus excite any movement of the body; but the common sensibility was not lost. The condition of cataleptic-like hypnotism lasted, altogether, about three hours and twenty minutes. Soon after this, some voluntary movements were made, and the rabbit gradually recovered to a perfectly normal state.

Any conclusion drawn from experiments on such animals as rabbits, with a substance whose predominating action is a soporific one, are always liable to objection. For this reason, we were induced to try the effect of iodide of methyl-morphium on man. One of us,[*] who is perfectly susceptible to the action of morphia, took on one occasion, half a grain of iodide of methyl-morphium, in the form of powder; but this produced no effect. On another occasion, one grain was taken, also as a powder; but not the slightest soporific or other action was caused. The latter dose contained about three-fourths of a grain of morphia, and this is certainly much above the usual narcotic dose of this substance.

It is important to mention, that although we have failed in causing any symptoms in warm-blooded animals with this substance, we have found that it acts with considerable energy on frogs. The nature of this action will be explained in the description of the effects of sulphate of methyl-morphium.

Sulphate of methyl-morphium ($(C_{17}H_{19}NO_3CH_3)_2SO_4$), was prepared by precipitating a solution of the iodide by means of sulphate of silver. It forms a white crystalline mass, very soluble in water. It gives the ordinary blue colour-reaction of morphia with persalts of iron.

This salt of methyl-morphium is much more active than the iodide. By subcutaneous injection, doses of two, three, four, five and eight grains caused marked symptoms; while a dose of ten grains was sufficient to kill a large rabbit. The effects of eight and of ten grains are described in the two following Experiments.

Experiment CXII.—Eight grains of sulphate of methyl-morphium, dissolved in distilled water, was injected under the skin, over the two flanks of a rabbit, weighing three pounds and one ounce. In twelve minutes, it appeared to be rather sleepy, and disinclined to move. In fourteen minutes, the head fell on the table, and the animal remained in this position, without any movements, except those that were necessary for respiration. In twenty-five minutes, the hypnotism was extremely well-marked; it was possible to place the animal in any position, and if this were compatible with stability, sound sleep occurred. A considerable stimulus was now required before the rabbit could be roused from sleep. In two hours and twenty minutes, this condition still continued, but the observations were now discontinued. On the following morning, the rabbit appeared to be perfectly well. No convulsive symptoms nor exaggeration of reflex activity was observed in this Experiment.

Experiment CXIII.—We dissolved ten grains of sulphate of methyl-morphium in 200 minims of distilled water, and injected the solution under the skin of a rabbit, weighing three pounds and eight ounces. In seven minutes, difficulty in moving about was observed; and, in rapid succession, some stumbles occurred, the limbs yielded, and the animal lay in a state of flaccidity, on the abdomen, chest, and lower jaw. It could now be placed without any resistance in almost any position. In twenty-four minutes, the respirations were very feeble and shallow, and at the rate of twenty-four in the minute; the rabbit was perfectly quiet and flaccid; and severe pinching of the skin excited only feeble reflex movements. There was not the slightest appearance of muscular rigidity, nor of starts, spasms, or even quivering movements. In forty-seven minutes, the respirations were extremely weak and jerky, and at the rate of ten per minute, while the sensibility of the conjunctiva and cornea had greatly diminished. In fifty-six minutes, the respirations occurred only eight times in the minute, and no movement of the eyelids could be excited by irritating the conjunctiva or cornea. Exophthalmos was now markedly present. Death occurred in one hour and two minutes after the administration of the poison.

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[*] Dr Fraser.

In the autopsy, the heart was found to be distended, and acting irregularly and slowly. There was no appearance of *rigor* two hours after death.

When administered by the stomach, twenty grains of sulphate of methylmorphium produced no effect on a rabbit.

Our experiments with morphia confirmed the observations made by others, which show that this alkaloid has two prominent actions on rabbits—a convulsant and a hypnotic one. We shall now consider how far each of these is modified by the addition of sulphate of methyl to morphia. The addition of iodide of methyl appears, no doubt, to have produced a very important change, but as this is rather in the direction of diminishing, or, as our experiments indicate, altogether destroying, the physiological activity of morphia, the iodide of methylmorphium may, in the mean time, be removed from consideration.

It has been proved, in a most satisfactory manner, that sulphate of methylmorphium possesses no convulsant action; for neither in the experiments we have described in detail, nor in any of the others we performed with this substance, was there any trace of spasmodic action or of exaggeration of the reflex function. It, however, undoubtedly causes hypnotic symptoms. In small non-fatal doses, hypnotism was chiefly manifested, and this rendered it somewhat difficult to judge whether paralysis were present or not. In large non-fatal doses, and in fatal doses, on the other hand, paralysis appeared to be the chief effect, though hypnotism was also present. It would, therefore, seem that sulphate of methylmorphium agrees with morphia in possessing a hypnotic action, but differs from it in producing paralysis, and in being free from all convulsant action. It is obvious that an objection might be urged against the latter part of this statement; for both the absence of convulsions and the production of paralysis might be merely the effects of hypnotism. Though we were ourselves convinced, from our experiments on rabbits, that such is not the case, we made some experiments on frogs to determine this more clearly.

EXPERIMENTS CXV. and CXX.—The blood-vessels were tied in one limb near the knee of two frogs, selected because of their resemblance to each other in weight and in activity. One grain of sulphate of methyl-morphium, in solution, was injected into the abdominal cavity of one of these frogs (*a*), and three-fourths of a grain of morphia, dissolved in very dilute sulphuric acid, into the abdominal cavity of the other (*b*).

(*a*). *Frog with Sulphate of Methyl-morphium.*

In eight minutes, the limbs yielded, and the frog subsided on the abdomen and chest.

In twenty minutes, it was perfectly flaccid, and the respirations had entirely ceased. Pinching of any portion of the skin excited energetic movements of the leg whose vessels were tied, and feeble movements in various other parts.

In thirty minutes, the two sciatic nerves were exposed; galvanism applied to their trunks caused contractions of the *tied limb, below the ligatures*, and nowhere else. The heart was now acting at the rate of forty-two in the minute, and the idio-muscular irritability was normal everywhere.

In twenty-four hours, the frog was still perfectly flaccid, the heart was contracting at the rate of thirty per minute, and the muscles of the poisoned and non-poisoned regions contracted when directly galvanised. Galvanism of the sciatic nerve of the poisoned leg, however, produced no movement; but galvanism of the sciatic nerve of the non-poisoned leg, even when applied to a part where the poison had access, still caused vigorous movements below the ligatures.

(*b*). *Frog with Sulphate of Morphia.*

In sixteen minutes, some slight sprawling occurred, before which the frog was jumping about vigorously.

In fifty minutes, pinching of the skin occasioned a series of clonic spasms, in which both posterior extremities were forcibly and slowly extended and then withdrawn, somewhat regularly, during three or four minutes, about four times in the minute. The movements then ceased, but they could be again excited.

In one hour, there was marked increase of the reflex excitability, a slight touch causing a spasmodic start.

In one hour and thirty-eight minutes, a slight touch of the skin excited a short tetanic convulsion.

In two hours, the same condition existed, and a tetanic convulsion could be at any time excited by a slight touch. During these convulsions, the muscles in the non-poisoned limb were contracted as forcibly as those in the poisoned regions.

In twenty-four hours, the frog was found dead, with all its muscles rigid.

These experiments prove distinctly that sulphate of methyl-morphium does not possess, in any degree, the convulsant action of morphia, but that it causes paralysis in place of convulsions. They also prove that this paralysis is due to an effect on the motor nerves. We have further determined, by the same method of experiment as has been already frequently described, that the peripheral terminations are the parts of the motor nerves which are primarily affected.

Iodide of methyl-morphium produces the same effects on frogs as sulphate, only a larger dose is required.

The poisonous activity of sulphate of methyl-morphium does not appear to be very different from that of a salt of morphia; for we have seen that for rabbits ten grains is about the minimum fatal dose of the former by subcutaneous injection, and this contains about eight grains of morphia, which is little above the fatal dose when subcutaneously exhibited. We have placed these and several other results, in a form convenient for comparison, in the following table.

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.
CI.	Iodide of methyl-morphium.	Rabbit, 2 lbs. 14 oz.	Subcutaneously.	20 grs. (containing 13·3 grs. of dry morphia).	No effect.
CV.	Morphia.	Do. (same rabbit as in Experiment CI.)	Subcutaneously.	8 grs. (containing 7·5 grs. of dry morphia).	Sleep in 1 hour and 4 minutes, epileptic convulsions in about 48 hours, and death some hours afterwards.
CVI.	Iodide of methyl-morphium.	Do., 3 lbs. 12 oz.	By stomach.	30 grs. (containing 20 grs. of dry morphia).	No effect.
CVII.	Morphia.	Do. (same rabbit as in Experiment CVI.)	By stomach.	5 grs. (containing 4·7 grs. of dry morphia).	Sleep in 1 hour and 6 minutes, and catalepsy in 1 hour and 25 minutes; these symptoms lasted for nearly 3 hours and 30 minutes, and were followed by recovery.
CXII.	Sulphate of methyl-morphium.	Do., 3 lbs. 1 oz.	Subcutaneously.	8 grs. (containing 6·6 grs. of dry morphia).	Sleep and partial paralysis in 14 minutes, continuing for more than 2 hours and 16 minutes, and followed by recovery.
CXIII.	Sulphate of methy-morphium.	Do., 3 lbs. 8 oz.	Subcutaneously.	10 grs. (containing 8·2 grs. of dry morphia).	Paralysis in 8 minutes, and doubtful sleepiness in 10 minutes; the paralysis became gradually more complete, and terminated in death, at 1 hour and 2 minutes after the administration of the poison.
CXXI.	Sulphate of methyl-morphium.	Do., 4 lbs. 3½ oz.	By stomach.	20 grs. (containing 16·4 grs. of dry morphia).	No effect.

NICOTIA.

The last substance in which we have now to describe the modifications produced by chemical addition is nicotia. This is a liquid alkaloid of great poisonous energy, derived from tobacco. It is a di-acid nitrile base, and has the formula $(C_{10}H_{14}N_2)$.

Iodide of methyl-nicotium.—Von Planta and Kekulé[*] investigated the action of iodide of ethyl on nicotia, and described a number of the ethyl-nicotium salts. The compounds of methyl-nicotium were investigated and described by Stahlschmidt.[†] When excess of iodide of methyl is added to nicotia, a considerable amount of heat is developed, and it is advisable to immerse the flask in which the mixture is made in cold water, in order to moderate the action; by this means the product (iodide of methyl-nicotium $(C_{10}H_{14}N_2(CH_3I)_2)$ is obtained nearly colourless, and crystallises almost as soon as it is cold. The crystalline powder is washed with a little cold alcohol, and crystallised from hot rectified spirit. Thus obtained, it forms tolerably large prismatic crystals, perfectly transparent and colourless, and free from the peculiar odour of nicotia. It is extremely soluble in water, so that for our purpose it was scarcely necessary to prepare the sulphate. More for the sake of symmetry, however, than because we expected to find any difference in action, we did so.

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[*] Annalen der Chemie und Pharmacie, vol. lxxxvii. p. 1 (1853).

[†] Ibid. vol. xc. p. 222 (1854).

A dose of five grains of iodide of methyl-nicotium, exhibited by subcutaneous injection, produced no effect on a rabbit. Ten grains caused trembling and slight impairment of motility; and the same symptoms occurred, in a somewhat exaggerated form, after the administration of fifteen grains: but recovery took place after both doses. The subcutaneous injection of twenty grains was followed, after several hours, by death. In the following account of the experiments in which fifteen and twenty grains were exhibited, it will be seen that no convulsive movements occurred during the progress of the symptoms.

EXPERIMENT CXXVII.—We injected fifteen grains of iodide of methyl-nicotium, dissolved in ninety minims of distilled water, into the subcutaneous cellular tissue of a rabbit, weighing three pounds. In eleven minutes, some trembling occurred, which, however, did not continue long; but it recurred in twenty-three minutes. In thirty minutes, it was observed that the head was supported with great difficulty, and shortly after it fell on the table, and the rabbit assumed a crouching attitude. There was no trembling so long as it was not disturbed; but whenever this was done, and when attempts were spontaneously made to assume some different position, the trembling recommenced. It continued in this condition for about an hour; soon afterwards the head was raised, and the trembling ceased. The rabbit was jumping about in a perfectly normal state two hours and three minutes after the administration.

EXPERIMENT CXXVIII.—We injected twenty grains of iodide of methyl-nicotium, dissolved in ninety minims of distilled water, into the subcutaneous cellular tissue of a rabbit, weighing about three pounds. In eight minutes, some trembling of the fore-legs was observed, which, however, soon ceased, and the rabbit sat down and remained quiet. In twenty minutes, the head fell upon the table, the neck muscle being apparently unable to support it; and in twenty-eight minutes, the paralysis had so far extended to the body that the rabbit, being unable to maintain even a crouching attitude, fell on the side. In one hour, it was in the flaccid condition of the last note, but the respiratory movements were few and feeble. In two hours and ten minutes, the respirations consisted of occasional gasps merely, and death appeared imminent. The observationswere unfortunately now (4 P.M.) interrupted until the following morning, when (10·15 A.M.) the rabbit was dead, and in *rigor mortis*.

In accordance with the plan followed in this investigation, we shall now describe the effects that are produced by nicotia itself,—and in order to obtain as exactly comparable data as possible, a portion of the nicotia used in the preparation of the iodide of methyl-nicotium employed in Experiments CXXVII. and CXXVIII. was administered to the rabbit which recovered from fifteen grains of the latter substance.

EXPERIMENT CXXXII.—One half-minim of nicotia (about 0·5 grain) was dissolved in fifteen minims of very dilute sulphuric acid, and the solution was injected into the subcutaneous cellular tissue of the rabbit employed, a week previously, in Experiment CXXVII. Symptoms were rapidly produced. In two minutes, spasmodic contractions occurred in the four limbs, which became extended, and raised the body in a convulsive manner. In three minutes, violent tremors occurred, and the whole body was convulsively agitated. In a few seconds afterwards, the limbs altogether yielded; the rabbit lay on the abdomen; and strong twitches occurred in the muscles of the neck, by which the head was jerked upwards, and in the limbs, by which the body was partially raised. This condition continued until ten minutes, when the spasmodic twitches ceased, and the rabbit fell on the side. It was now perfectly flaccid, with only twenty-five laboured respirations in the minute. In fourteen minutes, the respiratory movements were so feeble as to be scarcely visible; and, in fifteen minutes, they altogether ceased.

In the autopsy, the heart was found contracting, five minutes after death, at the rate of 160 per minute, but its contractions were feeble. The vermicular movements of the intestines appeared to be normal. The trunk of a sciatic nerve was irritated, ten minutes after death, and energetic movements followed in the limb to which the nerve was distributed.

Having found, in the case of iodide of methyl-nicotium, that so large doses of an extremely soluble substance were necessary to affect a rabbit by subcutaneous injection, we did not consider it advisable to determine how much was required to produce symptoms when it is exhibited by the stomach. For it may be almost positively asserted that, in the latter case, a much larger dose would be necessary; and while the administration of this would be inconvenient, because of its bulkiness, and of the difficulty of obtaining a large quantity in a perfectly pure form, the data obtained by subcutaneous injection are sufficient to prove the principal change that the addition of iodide of methyl produces in the physiological action of nicotia—namely, a great diminution in its poisonous activity.

Sulphate of methyl-nicotium ($C_{10}H_{14}N_2(CH_3)_2SO_4$) was prepared by precipitating a solution of the iodide by means of sulphate of silver. It forms a white, crystalline mass, extremely soluble in water.

On account of the readiness with which iodide of methyl-nicotium dissolves in water, it was not to be expected that any change in poisonous activity would be caused by its conversion into a sulphate; and the following experiment confirms this surmise, by showing that the activity of the sulphate is apparently no greater than that of the iodide.

Experiment CXXXIII.—Ten grains of sulphate of methyl-nicotium, dissolved in ninety minims of distilled water, was injected into the subcutaneous cellular tissue of a rabbit, weighing four pounds and three ounces. In ten minutes, some trembling occurred, accompanied with partial paralysis of the fore-legs. In twenty minutes, the head fell on the table, and, at intervals, series of tremors shook the whole body. It continued in this condition, the body being still supported by the legs, until fifty minutes, when ineffectual attempts were made to raise the head. These attempts were frequently repeated, and were finally successful at one hour and ten minutes; but the trembling, though now very slight, did not altogether cease until one hour and twenty minutes. After this, the rabbit seemed perfectly well.

In the absence of any very trustworthy or complete investigation into the mode in which nicotia acts, we cannot ascertain exactly how far its physiological properties are modified by chemical addition. It would appear, however, that the convulsive movements which are described as always occurring during nicotia poisoning, and which were well marked in Experiment CXXXII., are not among the symptoms produced by either iodide or sulphate of methyl-nicotium. The action of these substances is characterised by paralysis, accompanied with tremors, but unattended with spasms or convulsions. We performed the following experiments on frogs, in order to determine if this change were due not only to the disappearance of convulsive action, but also to the appearance of a paralysing action on motor nerves, similar to that so prominently possessed by the methyl derivatives of the other alkaloids examined in this paper.

Experiment CXXX.—The blood-vessels were tied in the left thigh of a frog, weighing 430 grains, and one grain of iodide of methyl-nicotium, dissolved in fifteen minims of distilled water, was then injected into the abdomen. In ten minutes, the anterior extremities had become so weak that they could not altogether support the thorax, but still the frog jumped about with considerable activity. In twenty-five minutes, the movements were sluggish, and the jumps were by no means so active as formerly, while some trailing of the posterior extremities was observed. The heart was acting at the rate of forty-two in the minute. In thirty-five minutes, irritation of any portion of the skin was followed by contractions of all the limbs, but these appeared to be rather more energetic in the left posterior (non-poisoned) limb than in the others. In forty minutes, the respirations were feeble, but the frog was sufficiently powerful to turn itself when placed on the back. In fifty-five minutes, severe pinching caused only slight reflex movements, of nearly equal strength, in both posterior extremities. In fifty-seven minutes, it was unable to turn when placed on the back, and the heart's contractions were at the rate of thirty-seven per minute. In one hour and thirty-nine minutes, irritation of any portion of the skin was followed by feeble, but nearly equal, movements of the four limbs. The observations were now interrupted. On the following morning the frog was jumping about normally.

EXPERIMENT CXXXI.—The blood-vessels were tied at the right knee of a frog, weighing 630 grains, and three grains of iodide of methyl-nicotium, dissolved in twenty minims of distilled water, was injected into the abdomen. In twenty-six minutes, the frog was lying, flaccid, on the abdomen and chest; and when the skin was irritated, reflex movements of equal strength were caused in the four limbs. In one hour and sixteen minutes, the flaccid state had become more marked, and, now, a somewhat stronger irritation was requisite in order to cause reflex movements, while these appeared to be of greatest strength in the right posterior (non-poisoned) limb. In two hours and forty-six minutes, the condition was exactly the same as last noted. The observations were now interrupted; and on the following morning the frog was found dead, and in *rigor*.

We obtained similar results with the sulphate.

It would, therefore, appear that though the convulsant effects of nicotia are not produced by its methyl derivatives, these derivatives do not possess any paralysing action on motor nerves. The change that is produced in the physiological action of nicotia is not the same as that which we have described in strychnia, brucia, thebaia, codeia, and morphia. We are inclined to believe, on account of this difference, that the convulsions of nicotia are not due to the same cause as in the other alkaloids we have examined.

A great diminution in physiological activity has, however, been produced by this chemical addition, and this will be at once recognised by referring to the following table:—

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.
CXXVII.	Iodide of methyl-nicotium.	Rabbit, 3 lbs.	Subcutaneously.	15 grs. (contain- 5·4 grs. of nicotia).	Trembling in 11 minutes, and partial paralysis in 30 minutes; these continued for about 49 and 50 minutes respectively, and a perfect recovery afterwards occurred.
CXXXII.	Nicotia (as sulphate).	Do. (same rabbit as in Experiment CXXVII.)	Subcutaneously.	0·5 min. (0.5 gr. nearly).	Convulsions in 3 minutes, and partial paralysis in less than 4 minutes; followed by death, 15 minutes after administration.
CXXXIII.	Sulphate of methyl-nicotium.	Do., 4 lbs. 3 oz.	Subcutaneously.	10 grs. (containing 5·6 grs. of nicotia).	Trembling in 10 minutes, slight paralysis in 20 minutes; perfect recovery in 1 hour and 20 minutes after administration.

Some experiments were made to determine, for our satisfaction, the physiological effects of iodide of methyl. The only bearing of these on the present investigation is, that no evidence was obtained in support of the extremely improbable hypothesis, that some of the changes produced in the action of the substances we have described might have been due to addition of the physiological action of the methyl compounds.

We have thus shown that chemical addition produces some important modifications in the action of those poisons which have been treated of in this communication. The action of strychnia, brucia, thebaia, codeia, morphia, and nicotia is evidently greatly diminished in degree, and, at the same time, strikingly changed in character.

The former effect is shown with all these alkaloids, especially when their action is compared with that of the iodides of their methyl derivatives. As all these iodides are much less soluble than the salts of the alkaloids themselves, it might be supposed that the diminution in activity could be explained by this difference in solubility. Some support is given to this supposition, by examining the relations between various of the substances included in this investigation. Thus, it has been demonstrated, on the one hand, that, for rabbits, the fatal dose of iodide of methyl-strychnium administered subcutaneously, is about twenty grains, and that of iodide of methyl-thebaium is about ten grains; while the former is soluble in 133 parts of distilled water, at a temperature of 37° C., and the latter in 16·5 parts at the same temperature. On the other hand, the fatal dose for rabbits, of sulphate of methyl-strychnium, is about four-fifths of a grain, and that of sulphate of methyl-thebaium is about five grains; while both substances are freely soluble, and with nearly equal readiness, in cold water. In these examples, the greater activity of strychnia over thebaia is manifested when a soluble salt of the methyl derivative of strychnia is employed; but when an extremely insoluble salt—the iodide—is employed, its activity is nearly the same as that of a corresponding preparation of thebaia; although the latter alkaloid is itself considerably less energetic than strychnia. It is, therefore, apparent that poisonous activity may be modified by the degree of solubility,—a well-recognised principle in toxicological physiology. But while the diminished activity of the iodides of many of these methyl derivatives may be greatly due to the difficulty of dissolving them, this explanation is inapplicable to iodide of methyl-nicotium,—an extremely soluble substance,—and it is insufficient to account for the differences of activity between the majority of the sulphates of the methyl derivatives and the salts of the alkaloids themselves. Our investigation has not furnished us with any explanation of the change in these sulphates. There are several possible explanations, but we shall not specially allude to them, as their discussion can only be properly undertaken after experimental examination of a laborious and difficult nature, and but indirectly connected with our present subject. When, however, we compare the activity of the sulphates of the majority of the ammonium bases considered in this paper with that of the corresponding iodides, we observe striking differences, which cannot be explained by differences of solubility alone, but which, we believe, must be also due to the remarkable stability possessed by these iodides. Strychnia is a much less soluble substance than iodide of methyl-strychnium, and yet a rabbit that survived the administration of fifteen grains of iodide of methyl-strychium, was killed in a few minutes by the administration of one-twentieth of a grain of strychnia. Before absorption, the strychnia may have been converted into a more soluble form, and this change may have facilitated its absorption, and permitted it to be carried by the blood-stream to the tissues it affects; but the great *stability* of the iodide of methyl-strychnium prevents its conversion into a more soluble form, and so impedes greatly the absorption. Just as in the more familiar case of the salts of lead, the sulphate is inert while the carbonate is poisonous, although they are both insoluble; and this difference of physiological action is undoubtedly due to the fact, that the carbonate, on account of its instability, is readily converted in the stomach into a soluble salt, while no such change takes place in the case of the sulphate. Stability may also influence the physiological activity of these iodides, even after their absorption, by preventing those chemical actions on the tissues by which many of the effects of poisons are probably caused.

The change in the *character* of the physiological action is remarkably illustrated by strychnia, brucia, and thebaia, whose purely spinal-stimulant action is converted into a paralysing action on the periphery (end-organs) of motor nerves; it is apparent in codeia and morphia, whose convulsant action is also converted into a paralysing action on motor nerve end-organs, and whose hypnotic action is apparently altogether destroyed in the case of codeia, and certainly greatly diminished in that of morphia; and it is obviously, though less so than with the others, in the case of nicotia, whose convulsant action is diminished if not altogether removed. We may conclude from these facts, that when a nitrile base possesses a strychnia-like action, the salts of the corresponding ammonium bases have an action identical with that of curare.

It is well known that curare and strychnia are derived from plants belonging to the same *genus*, and it is, therefore, interesting to observe such a relationship. It may not, however, be altogether superfluous to add, that strychnia, brucia, and the other spinal-stimulant alkaloids examined in this paper, have not been converted by chemical addition into curarina,—the active principle of curare. The action of the methyl derivatives of these bases is of precisely the same character as that of curare, and they possess the same peculiarity of slow absorption by the mucous membrane of the digestive system, but the degrees of their activity are very different. If we confine our attention to the salts of the methyl derivatives of strychnia, brucia, and thebaia, where the action is uncomplicated, we observe that they form a series in which the fatal dose varies for each, while this dose in the case of the most active of the three is considerably above that of curare, and greatly above that of curarina. Besides, curarina has a characteristic colour reaction that belongs to none of these bodies; and the latter further prove this dissimilarity by each of them possessing special colour reactions by which they may be distinguished from each other.

It is not only of great interest, but probably of some practical value, that five new compounds should be found having the physiological action of curare. The great difficulty of obtaining this substance has hitherto proved a serious barrier to its therapeutical employment. Although none of the compounds that we have shown to act as curare does are so energetic as that substance, three of them—sulphate of methyl-strychnium, sulphate of methyl-brucium, and sulphate of methyl-thebaium—are sufficiently so to fulfil all possible therapuetical requirements, and even to rank as powerful poisons. Moreover, they may be readily obtained in a state of perfect purity, and, therefore, of constant strength; and, in this respect, they possess a great advantage over curare.

The six alkaloids we have examined may be divided into two classes, according to the readiness with which they combine with iodide of methyl. The one class includes strychnia, brucia, thebaia, and nicotia; and the other, codeia and morphia; and the combination is much more easy with the former than with the latter class. Without attaching any general significance to the occurrence, it may not be altogether unworthy of being pointed out that in our experience, therefore, the more active poisons are the more readily acted upon by iodide of methyl.

It is curious, though not unexpected, that the ordinary colour reactions of the alkaloids are retained by their methyl derivatives. This may possibly prove of some importance to the medical jurist; and as these compounds are not precipitated by alkalies, nor by the carbonates of the alkalies, some difficulty may be met with in discovering their presence in cases of poisoning.

TABULAR SUMMARY OF EXPERIMENTS.

No of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.	Interval between administration and commencement of symptoms.	Duration of symptoms.	Notes.
I.	Iodide of methyl-strychnium.	Rabbit, 3 lbs.	Subcutaneously.	5 grs.	None.			
II.	Do.	Do., 3 lbs. 14 oz.	Do.	6 grs.	None.			
* III.	Do.	Do., 3 lbs. 12 oz.	Do.	8 grs.	None.			
IV.	Do.	Do., 3 lbs. 4 oz.	Do.	8 grs.	None.	...	...	Same rabbit, on three consecutive days.
V.	Do.	Do., 3 lbs. 4 oz.	Do.	10 grs.	None.			
VI.	Do.	Do., 3 lbs. 4 oz.	Do.	12 grs.	None.			
VII.	Do.	Do., 3 lbs. 8 oz.	Do.	15 grs.	Paralysis followed by recovery.	45 minutes.	More than 2 hours.	
VIII.	Do.	Do., 3 lbs. 2½ oz.	Do.	20 grs.	Death, preceded by paralysis.	20 minutes.	50 minutes.	
IX.	Do.	Cat (full grown).	Do.	1 gr.	None.			
X.	Do.	Do. "	Do.	5 grs.	None.			
XI.	Do.	Frog, 541 grs.	Do.	1 gr.	Paralysis.	11 minutes.	Notobserved.	
* XII.	Do.	Do., 410 grs.	Do.	0·5 gr.	Paralysis.	12 minutes.	Notobserved.	
XIII.	Strychnia (suspended in distilled water).	Rabbit, 3 lbs. 8 oz.	Do.	0·05 gr.	Death, preceded by spasms and numerous tetanic convulsions.	8 minutes.	22 minutes.	Same rabbit as survived 15 grs. of iodide of methyl-strychnium (Experiment VII.)
XIV.	Iodide of methy-strychnium.	Rabbit, 2 lbs. 10 oz.	By stomach.	12 grs.	None.	...	...	Same rabbit. Experiment XIV. was made 48 hours after Experiment XV.
XV.	Do.	Do., 2 lbs. 10 oz.	Do.	15 grs.	None.			
XVI.	Do.	Do., 3 lbs. 2 oz.	Do.	20 grs.	None.			
XVII.	Do.	Do., 3 lbs. 13 oz.	Do.	30 grs.	None.			
XVIII.	Strychnia (as hydrochlorate).	Do.. 3 lbs. 2 oz.	Do.	0·075 gr.	Death, preceded by numerous spasms and tenanic convulsions.	Exaggeration of reflux function in 24 minutes.	37 minutes.	Same rabbit as survived 20 grs. of iodide of methyl-strychnium (Experiment XVI.)

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* Several of the experiments included in this Table were performed after the reading of the paper. They are distinguished by an asterisk; and it will be observed that the details of a few of them have been incorporated in the text.

TABULAR SUMMARY OF EXPERIMENTS—*continued.*

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.	Interval between administration and commencement of symptoms.	Duration of symptoms.	Notes.
XIX.	Strychnia (as hydrochlorate).	Rabbit, 3 lbs. 13 oz.	By stomach.	0·1 gr.	Death, preceded by numerous spasms and tetanic convulsions.	17 minutes.	24 minutes.	Same rabbit as survived 30 grs. of iodide of methyl - strychnium (Experiment XVII.)
XX.	Sulphate of methyl-strychnium.	Do., 2 lbs. 5 oz.	Subcutaneously.	0·1 gr.	None.			
XXI.	Do.	Do., 3 lbs. 3½ oz.	Do.	0·2 gr.	None.			
XXII.	Do.	Do., 2 lbs. 5 oz.	Do.	0·5 gr.	None.			
XXIII.	Do.	Do., 3 lbs. 3½ oz.	Do.	0·8 gr.	Paralysis, followed by recovery.	28 minutes.	54 minutes.	
XXIV.	Do.	Do., 3 lbs. 1 oz.	Do.	0·8 gr.	Death, preceded by paralysis.	Paralysis in 9 minutes.	21 minutes.	Some doubt of the absolute purity of the sulphate used in this Experiment.
XXV.	Do.	Do., 2 lbs. 14 oz.	Do.	1 gr.	Do. do.	11 minutes.	7 minutes.	
XXVI.	Do.	Do., 3 lbs. 10 oz.	Do.	5 grs.	Do. do.	13 minutes.	4 minutes.	
XXVII.	Do.	Dog, 40 lbs. (?).	Do.	12 grs.	Do. do.	Partial paralysis in 4 minutes.	15 minutes.	
XXVIII.	Do.	Frog, 820 grs.	Do.	0·1 gr.	Paralysis.	4 minutes.	Not observed.	
* XXIX.	Do.	Do., 615 grs.	Do.	0·1 gr.	Paralysis.	3 minutes.	Not observed.	
* XXX.	Do.	Rabbit, 3 lbs. 3 oz.	By stomach	2 grs.	None.	...	...	Same rabbit. Experiments XXX. and XXXI. were performed after an interval of 48 hours, and Experiments XXXI. and XXXII. after one of 24 hours.
* XXXI.	Do.	Do., 3 lbs. 3 oz.	Do.	5 grs.	None.			
* XXXII.	Do.	Do., 3 lbs. 3 oz.	Do.	10 grs.	None.			
XXXIII.	Do.	Do., 3 lbs. 5¾ oz.	Do.	20 grs.	None.			
* XXXIV.	Do.	Do., 3 lbs. 3 oz.	Do.	25 grs.	Death, preceded by tremors and paralysis.	55 minutes.	More than 2 hours.	

TABULAR SUMMARY OF EXPERIMENTS—*continued.*

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.	Interval between administration and commencement of symptoms.	Duration of symptoms.	Notes.
XXXV.	Nitrate of methyl-strychnium.	Rabbit, 4 lbs. 1 oz.	Subcutaneously.	10 grs.	Death, preceded by tremors and paralysis.	6 minutes.	4½ minutes.	
XXXVI.	Hydrochlorate of ethyl-strychnium.	Frog, 240 grs.	Placed on skin.	0·5 gr. (?).	Paralysis, followed by recovery.	2 minutes and 30 sec.	More than 4 hours.	
XXXVII.	Do.	Do., 400 grs.	Subcutaneously.	0·5 gr.	Death, preceded by paralysis.	2 minutes.	About 15 ho.	
XXXVIII.	Iodide of methyl-brucium.	Rabbit, 3 lbs. 11 oz.	Do.	8 grs.	None.			
XXXIX.	Do.	Do., 3 lbs. 11 oz.	Do.	12 grs.	None.			
XL.	Do.	Do., 4 lbs.	Do.	15 grs.	Paralysis, followed by recovery.	2 hours and 43 minutes.	More than 1 hour and 15 minutes.	
XLI.	Do.	Do., 3 lbs. 12 oz.	Do.	18 grs.	Death, preceded by paralysis.	27 minutes.	46 minutes.	
XLII.	Brucia (as hydrochlorate).	Do., 4 lbs.	Do.	0·1 gr.	None.			
XLIII.	Do.	Do., 4 lbs.	Do.	0·2 gr.	Death, preceded by numerous spasms and tetanic convulsions.	7 minutes.	11 minutes and 30 sec.	Same rabbit as survived 15 grs. of iodide of methyl-brucium (Experiments XL.)
XLIV.	Iodide of methyl-brucium.	Frog, 380 grs.	Do.	0·5 gr.	Death, preceded by paralysis.	16 minutes.	Not observed.	
XLV.	Do.	Rabbit, 3 lbs.	By stomach.	20 grs.	None.			
XLVI.	Do.	Do., 4 lbs. 2 oz.	Do.	30 grs.	None.			
* XLVII.	Brucia (as hydrochlorate.	Do., 3 lbs.	Do.	0·2 gr.	None.			
* XLVIII.	Do.	Do., 2 lbs. 1 oz.	Do.	0·3 gr.	None.			
* XLIX.	Do.	Do., 3 lbs. 11 oz.	Do.	0·5 gr.	None.			
* L.	Do.	Do., 2 lbs. 13 oz.	Do.	1 gr.	None.			
LI.	Do.	Do., lbs. 2 oz.	Do.	2 grs.	Death, proceded by spasms and tetanic convulsions.	40 minutes.	More than 2 hours and 10 minutes.	Same rabbit as survived 30 grs. of iodide of methyl-brucium (Experiment XLVI.)

TABULAR SUMMARY OF EXPERIMENTS—*continued.*

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.	Interval between administration and commencement of symptoms.	Duration of symptoms.	Notes.
LII.	Sulphate of methyl-brucium.	Rabbit, 2 lbs. 4 oz.	Subcutaneously.	1 gr.	None.			
LIII.	Do.	Do., 2 lbs. 13¾ oz.	Do.	2 grs.	Paralysis, followed by recovery.	10 minutes.	About 1 hour and 50 min.	
LIV.	Do.	Do., 3 lbs. 14½ oz.	Do.	2·5 grs.	Death, preceded by paralysis.	20 minutes.	50 minutes.	
* LV.	Do.	Do., 3 lbs. 5 oz.	Do.	3 grs.	Do. do.	12 minutes.	16 min. and 15 seconds.	
LVI.	Sulphate of methyl-brucium.	Frog., 608 grs.	Injected into abdomen.	0·2 gr.	Do. do.	3 to 4 minutes.	Not observed.	
* LVII.	Do.	Do., 542 grs.	Do.	0·16 gr.	Do. do.	6 minutes.	Not observed.	
LVIII.	Do.	Rabbit, 4 lbs. 2 oz.	By stomach.	20 grs.	None.			
LIX.	Iodide of methyl-thebaium.	Do., 2 lbs. 12 oz.	Subcutaneously.	1 gr.	None.	...	...	Same rabbit; intervals of at least 24 hours elapsed between each experiment.
LX.	Do.	Do., 2 lbs. 12 oz.	Do.	5 grs.	None.	...	...	
LXI.	Do.	Do., 2 lbs. 12 oz.	Do.	6 grs.	None.	...	...	
LXII.	Do.	Do., 2 lbs. 12 oz.	Do.	8 grs.	Paralysis, followed by recovery.	30 minutes.	1 hour and 30 minutes.	
LXIII.	Do.	Do., 2 lbs. 11 oz.	Do.	10 grs.	Death, preceded by paralysis.	10 minutes.	13 minutes and 30 sec.	
* LXIV.	Do.	Do., 2 lbs. 15 oz.	Do.	12 grs.	Do. do.	Less than 11 minutes.	10 minutes nearly.	
LXV.	Thebaia (as hydrochlorate).	Do., 2 lbs. 12 oz.	Do.	0·1 gr.	Slight exaggeration of the reflex function.	About 50 minutes.	About 1 hour.	Same rabbit as survived 8 grs. of iodide of methyl-thebaium (Experiment LXII.)
LXVI.	Do.	Do., 2 lbs. 12 oz.	Do.	0·2 gr.	Death, preceded by spasms and tetanic convulsions.	40 minutes.	14 minutes.	
LXVII.	Do.	Do., 2 lbs. 14 oz.	Do.	0·2 gr.	Do. do.	33 minutes.	1 hour, 12 minutes and 44 seconds.	
LXVIII.	Iodide of methyl-thebaium.	Frog, 460 grs.	Do.	0·5 gr.	Death, preceded by paralysis.	12 minutes.	Not observed.	
* LXIX.	Do.	Do., 540 grs.	Do.	0·5 gr.	Do. do.	16 minutes.	Not observed.	

TABULAR SUMMARY OF EXPERIMENTS—*continued.*

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.	Interval between administration and commencement of symptoms.	Duration of Symptoms.	Notes.
* LXX.	Iodide of methyl-thebaium.	Rabbit, 3 lbs.	By stomach.	10 grs.	None.			
LXXI.	Do.	Do., 4 lbs. 6 oz.	Do.	30 grs.	None.			
LXXII.	Thebaia (as hydrochlorate).	Do., 4 lbs. 6 oz.	Do.	3 grs.	Spasms, followed by recovery.	16 minutes.	More than 1 hour.	Same rabbit as survived 30 grs. of iodide of methyl-thebaium (Experiment LXXI.) The experiments were made at intervals of at least 48 hours.
LXXIII.	Do.	Do., 4 lbs. 6 oz.	Do.	3·5 grs.	Spasmodic convulsions, followed by recovery.	14 minutes.	More than 2 hours.	
LXXIV.	Do.	Do., 4 lbs. 6 oz.	Do.	4 grs.	Death, preceded by numerous tetanic convulsions.	4 minutes.	15 minutes.	
LXXV.	Sulphate of methyl-thebaium.	Do., 2 lbs. 14 oz.	Subcutaneously.	2 grs.	None.			
LXXVI.	Do.	Do., 3 lbs. 15 oz.	Do.	4 grs.	Slight increase of reflex activity, followed by recovery.	22 minutes.	15 minutes.	Faintly coloured preparation.
* LXXVII.	Do.	Do., 3 lbs. 11½ oz.	Do.	4·5 grs.	Paralysis, followed by recovery.	15 minutes.	About 2 hours and 15 min.	Pure preparation.
LXXVIII.	Do.	Do., 4 lbs. 0½ oz.	Do.	5 grs.	Death, preceded by paralysis.	13 minutes.	37 minutes.	
* LXXIX.	Do.	Frog, 420 grs.	Injected into abdomen.	0·2 gr.	Do. do.	5 minutes.	Not observed.	
* LXXX.	Do.	Do., 604 grs.	Do.	0·2 gr.	Do. do.	7 minutes.	Not observed.	
* LXXXI.	Do.	Rabbit, 4 lbs. 4 oz.	By stomach.	20 grs.	None.			
LXXXII.	Iodide of methyl-codeium.	Do., 3 lbs. 1 oz.	Subcutaneously.	5 grs.	None.			
LXXXIII.	Do.	Do., 3 lbs.	Do.	10 grs.	Paralysis, followed by recovery.	27 minutes.	More than 2 hours.	
LXXXIV.	Do.	Do., 2 lbs. 14 oz.	Do.	15 grs.	Do. do.	22 min. and 20 seconds.	4 hours and 3 minutes.	
* LXXXV.	Do.	Do., 2 lbs. 12½ oz.	Do.	20 grs.	Death, preceded by paralysis.	13 minutes.	19 minutes.	

TABULAR SUMMARY OF EXPERIMENTS—*continued.*

No. of Experiments.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.	Interval between administration and commencement of symptoms.	Duration of Symptoms.	Notes.
LXXXVI.	Iodide of methyl-codeium.	Frog, 701 grs.	Subcutaneously.	2 grs.	Death, preceded by paralysis.	21 minutes.	Not observed.	
*LXXXVII.	Do.	Do., 610 grs.	Do.	2·5 grs.	Do. do.	11 minutes.	Not observed.	
LXXXVIII.	Codeia.	Rabbit, 3 lbs.	Do.	0·5 gr.	Abnormal frequency of respiration, increase of reflex activity, and slight hypnotism, followed by recovery.	14 min. and 30 seconds.	About 4 hours.	
LXXXIX.	Do.	Do., 2 lbs. 14 oz.	Do.	1 gr.	Death, preceded by spontaneous twitches, and numerous tetanic convulsions.	15 minutes.	1 hour and 30 minutes.	Same rabbit as survived 15 grs. of iodide of methyl-codium (Experiment LXXXIV.)
XC.	Iodide of methyl-codeium.	Do., 3 lbs. 12 oz.	By stomach.	15 grs.	None.			
XCI.	Do.	Do., 2 lbs. 13 oz.	Do.	30 grs.	None.			
XCII.	Codeia (as hydrochlorate).	Do., 2 lbs. 5 oz.	Do.	5 grs.	None.			
* XCIII.	Do.	Do., 2 lbs. 5 oz.	Do.	7 grs.	Hypnotism, and increase of reflex cavity, followed by recovery.	43 minutes.	More than 2 hours and 30 seconds.	
* XCIV.	Do.	Do., 3 lbs. 13 oz.	Do.	10 grs.	Do. do.	24 minutes.	More than 2 hours and 36 minutes.	Same rabbit as received into the stomach, without any effect, 30 grs. of iodide of methyl-codeium (Experiment XCI.)
XCV.	Sulphate of methyl-codeium.	Do., 3 lbs. 2 oz.	Subcutaneously.	4 grs.	None.			
* XCVI.	Do.	Do., 4 lbs.	Do.	8 grs.	Paralysis, followed by recovery.	30 minutes.	About 1 hour and 20 min.	No hypnotism was observed in either of these experiments.
XCVII.	Do.	Do., 4 lbs. 4 oz.	Do.	10 grs.	Death, preceded by paralysis.	23 minutes.	19 minutes.	

TABULAR SUMMARY OF EXPERIMENTS—*continued.*

No. of Experiments.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.	Interval between administration and commencement of symptoms.	Duration of Symptoms.	Notes.
XCVIII.	Sulphate of methyl-codeium.	Frog, 722 grs.	Injected into abdomen.	1 gr.	Death preceded by paralysis.	14 minutes.	Not observed.	
* XCIX.	Do.	Do., 694 grs.	Do.	1 gr.	Do. do.	11 minutes.	Not observed.	
C.	Do.	Rabbit, 4 lbs. 13 oz.	By stomach.	20 grs.	None.			
CI.	Iodide of methyl-morphium.	Do., 2 lbs. 14 oz.	Subcutaneously.	20 grs.	None.			
CII.	Do.	Do., 2 lbs.	Do.	20 grs.	None.			
* CIII.	Do.	Frog, 501 grs.	Do.	1 gr.	Paralysis, followed by recovery.	1 hour and 25 minutes.	More than 24 hours.	
* CIV.	Do.	Do., 410 grs.	Do.	2 grs.	Death, preceded by paralysis.	50 minutes.	Not observed.	
CV.	Morphia.	Rabbit, 2 lbs. 14 oz.	Do.	8 grs.	Death, preceded by hypnotism and epileptic convulsions.	1 hour and 4 minutes.	More than 48 hours.	Same rabbit as survived 20 grs. of iodide of methyl-morphium (Experiment CI.)
CVI.	Iodide of methyl-morphium.	Do., 3 lbs. 12 oz.	By stomach.	30 grs.	None.			
CVII.	Morphia.	Do., 3 lbs. 12 oz.	Do.	5 grs.	Hyptonism and catalepsy, followed by recovery.	1 hour and 6 minutes.	About 3 hours and 30 min.	Same rabbit as received 30 grs. of iodide of methyl-morphium (Experiment CVI.)
CVIII.	Sulphate of methyl-morphium.	Do., 3 lbs.	Subcutaneously.	2 grs.	Hypnotism (and paralysis?), followed by recovery.	54 minutes.	More than 3 hours.	
* CIX.	Do.	Do., 3 lbs. 5 oz.	Do.	3 grs.	Do. do.	15 minutes.	More than 2 hours.	
* CX.	Do.	Do., 3 lbs. 1oz.	Do.	4 grs.	Do. do.	18 minutes.	More than 3 hours.	
* CXI.	Do.	Do., 3 lbs. 8 oz.	Do.	5 grs.	Do. do.	15 minutes.	More than 3 hours.	
CXII.	Do.	Do., 3 lbs. 1 oz.	Do.	8 grs.	Hypnotism and paralysis, followed by recovery.	12 minutes.	More than 2 hours.	

TABULAR SUMMARY OF EXPERIMENTS—*continued.*

No. of Experiments.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.	Interval between administration and commencement of symptoms.	Duration of Symptoms.	Notes.
CXIII.	Sulphate of methyl-morphium.	Rabbit, 3 lbs. 8 oz.	Subcutaneously.	10 grs.	Death, preceded by paralysis.	7 minutes.	55 minutes.	
* CXIV.	Do.	Do., 2 lbs. 13 oz.	Do.	15 grs.	Do. do.	5 minutes.	7 minutes.	
CXV.	Do.	Frog, 480 grs.	Injected into abdominal cavity.	1 gr.	Do. do.	8 minutes.	Not observed.	
* CXVI.	Do.	Do., 444 grs.	Do.	1 gr.	Do. do.	4 minutes.	Not observed.	
CXVII.	Morphia (as sulphate).	Rabbit, 3 lbs. 8½ oz.	Subcutaneously.	2 grs.	Hypnotism, followed by recovery.	20 minutes.	More than 3 and less than 24 hours.	
CXVIII.	Do.	Do., 3 lbs. 2 oz.	Do.	4 grs.	Do. do.	7 minutes.	More than 48 and less than 72 hours.	
CXIX.	Do.	Do., 3 lbs. 7 oz.	Do.	5 grs.	Do. do.	7 minutes.	Not observed.	
CXX.	Do.	Frog, 510 grs.	Injected into abdominal cavity.	0·75 gr.	Death, preceded by increase of the reflex function, spasms, tetanic convulsions, and partial paralysis.	16 minutes.	More than 1 hour and 44 min. and less than 24 hours.	
CXXI.	Sulphate of methyl-morphium.	Rabbit, 4 lbs. 3½ oz.	By stomach.	20 grs.	None.			
CXXII.	Nitrate of methyl-morphium.	Do., 3 lbs. 2 oz.	Subcutaneously.	20 grs.	Death, preceded by tremors and paralysis.	3 minutes and 30 seconds.	10 minutes.	
CXXIII.	Iodide of methyl-nicotium.	Do., 3 lbs. 7 oz.	Do.	5 grs.	None.			
* CXXIV.	Do.	Do., 2 lbs. 3 oz.	Do.	5 grs.	None.			
CXXV.	Do.	Do., 3 lbs. 4 oz.	Do.	10 grs.	Tremors and partial paralysis, followed by recovery.	10 minutes.	1 hour and 18 minutes.	Not an absolutely pure preparation; the colour being yellowish brown.
CXXVI.	Do.	Do. 2 lbs. 10 oz.	Do.	10 grs.	Slight tremors, followed by recovery.	13 minutes.	About 20 min.	Pure, nearly colourless and odourless preparation.

TABULAR SUMMARY OF EXPERIMENTS—*continued.*

No. of Experiment.	Substance employed.	Animal and its weight.	Method of exhibition.	Dose.	Effect.	Interval between administration and commencement of symptoms.	Duration of symptoms.	Notes.
CXXVII.	Iodide of methyl-nicotium.	Rabbit, 3 lbs.	Subcutaneously.	15 grs.	Tremors and partial paralysis, followed by recovery.	11 minutes.	49 minutes.	
CXXVIII.	Do.	Do., 3 lbs.	Do.	20 grs.	Death, preceded by tremors and paralysis.	8 minutes.	About 2 hours.	
* CXXIX.	Do.	Do., 3 lbs. 2 oz.	Do.	25 grs.	Do. do.	6 minutes.	1 hour and 38 minutes.	
CXXX.	Do.	Frog, 430 grs.	Injected into abdominal cavity.	1 gr.	Partial paralysis, followed by recovery.	15 minutes.	More than 1 and 30 min. and less than 15 hours.	
* CXXXI.	Do.	Do., 630 grs.	Do.	3 grs.	Death, preceded by paralysis.	10 minutes.	More than 2 and 26 min. and less than 20 hours.	
CXXXII.	Nicotia (as sulphate).	Rabbit, 3 lbs.	Subcutaneously.	6·5 minim (=about 0·5 gr.).	Death, preceded by tremors, violent convulsions, and partial paralysis.	2 minutes.	13 minutes.	Same rabbit as survived 15 grs. of iodide of methyl-nicotium (Experiment CXXVII.)
CXXXIII.	Sulphate of methyl-nicotium.	Do., 4 lbs. 3 oz.	Do.	10 grs.	Slight tremors, and partial paralysis, followed by recovery.	10 minutes.	1 hour and 10 minutes.	
CXXXIV.	Do.	Frog, 683 grs.	Injected into abdominal cavity.	3 grs.	Death, preceded by paralysis.	32 minutes.	More than 2 and less than 48 hours.	

VI.—*On the Products of the Destructive Distillation of Animal Substances.* Part V. By Thomas Anderson, M.D., Professor of Chemistry in the University of Glasgow.

(Read 20th April 1868.—Sent for publication, November 1868).

In the fourth part of these researches, I described a new base produced by the action of sodium upon picoline, to which I gave the name of Parapicoline, because it has the same composition as picoline, although the circumstances of its formation appeared to show that it had been produced by the combination of two molecules of that substance, so that its true formula would be $C_{12}H_{14}N_2$. Unfortunately, its high boiling point, and tendency to decompose when distilled, made it impossible to determine its vapour density, which afforded the only means of ascertaining whether this hypothesis was correct; and it was only assumed, because similar cases of polymerisation had been established beyond a doubt in the case of other classes of organic compounds. In the hope of obtaining a similar base of lower boiling point, and therefore better adapted to the necessary experiments, I have submitted pyridine to the action of sodium, and the results of the inquiry are contained in the following pages.

My earlier experiments were conducted in precisely the same manner as those with picoline. Dry pyridine was heated to its boiling point along with small pieces of sodium, amounting to about one-fifth of its weight, in a flask furnished with a long cohobating tube. As the temperature rose, the pieces of sodium became covered with a brown coating; purple streaks appeared in the fluid, which, however, soon disappeared again; and after some hours the whole fluid was converted into a dark-brown or black mass, which was viscous when hot, and on cooling solidified into a hard brittle resin. In this a few white powdery nodules are disseminated, which explode violently when brought in contact with water. A large portion of the sodium employed remains unacted on, and if the operation has been well performed, is generally found collected into one or two large pieces, which can be easily separated from the resinous mass. After the sodium has been removed as thoroughly as possible, the crude product is thrown into water in small successive portions, so as to avoid the risk of explosions from any particles of sodium which may have remained disseminated through it. The water soon becomes alkaline, owing to the presence of caustic soda; unchanged pyridine makes itself manifest by its powerful smell, and the resin is slowly converted into a thick, viscid oil of dark-brown colour, and nearly, or altogether, insoluble in water, which collects at the bottom of the vessel. This oil is washed several times with water, dried over calcic-chloride, and distilled. The distillation is best effected in a current of hydrogen, and at a temperature below the boiling point of the oil. A little unchanged pyridine distils at first, accompanied by a small quantity of a light oil, insoluble in water, having a pungent smell similar to, and yet appreciably different from, that of pyridine; and which, as we shall afterwards see, appears to be a mixture of several bases. As the temperature rises, a thick, heavy, and yellowish oil, having a peculiar smell, in no degree pungent, but dull, heavy, and somewhat resembling that of soot, passes over. As the distillation proceeds, crystals make their appearance in the neck of the retort. At a certain stage of the process the product becomes nearly solid, and on cooling, crystals are deposited from the fluid distillate. Towards the close of the distillation some ammonia and very volatile bases are evolved, obviously produced by the decomposition of the oil passing over; and a dark resinous mass remains in the retort, which can be forced over by raising the temperature, in doing which a large part of it is decomposed, and a residue of charcoal is left in the retort.

The products of the action being obviously complicated, the whole was cautiously redistilled, and the portion which solidified in the neck of the retort collected apart, while the fluid portions having been introduced into a freezing mixture of snow and salt, soon gave an abundant crop of crystals. These were purified by pressure between folds of filtering paper, and crystallisation from water or alcohol, in both of which they are soluble, until they have lost the smell of the oil by which they are accompanied.

After having proceeded some way in the investigation, I found that the same substances could be obtained with greater certainty by a modification of the process just described. It is by no means necessary to heat the sodium and pyridine together, for the action takes place in the cold; but in this case it is slower, and the phenomena are somewhat different. The brown appearance on the surface of the sodium and the purple streaks appear in the fluid at the beginning of the action, but the pyridine does not become brown, it retains its colour, and the sodium is covered with a black crust, which, after two or three days, exceeds it in bulk, is quite brittle, and sometimes shows a tendency to separate into layers. The pyridine acquires a yellowish tint, and then contains in solution an oil insoluble in water. When the action is judged to have gone sufficiently far, the sodium with its crust is removed from the fluid and washed with a small quantity of pure pyridine, so as to get rid of any of the oily base which may remain attached to it. The crust is then detached as thoroughly as possible from the sodium and thrown into water, any sodium still adhering to it burns, and a dark gray, almost black, powder falls to the bottom of the glass. This is washed first by decantation, and afterwards on a cloth filter until it is free from soda, and on being opened out and exposed for some time to the air, it is entirely converted into a snow-white mass of interlaced acicular crystals identical with those obtained by the first process.

DIPYRIDINE.

The substance so obtained is a base to which, for reasons that will be immediately apparent, I give the name of dipyridine. It forms white crystals fusing at 108° Cent., and solidifies on cooling into a crystalline mass. It volatilises slowly at 100°, and sublimes unchanged at a high temperature, giving long needle-shaped crystals. It is rather sparingly soluble in cold but readily in boiling water, and the fluid, on cooling, becomes filled with a mass of interlaced needles. It is readily soluble in alcohol and ether, and the boiling solutions give acicular crystals on cooling. It likewise dissolves in pyridine and in volatile oils. From the oily base along with which it distils in the first process of preparation, it crystallises in short, thick, four-sided prisms, which are transparent so long as they remain in the fluid, but soon become opaque when they are removed from it. When well purified they are inodorous, but in general they have a faint smell, due, apparently, to a trace of the fluid base adhering to them. Dipyridine is a very stable compound. It is not decomposed by hydrochloric, sulphuric, or nitric acids. Potash and ammonia precipitate it from its solutions in acids as a mass of minute crystals. Its aqueous solution gives no precipitates with solutions of sulphate of magnesia, zinc, nickel, acetate of lead, or perchloride of iron. With sulphate of copper it gives a pale bluish-white precipitate, with corrosive sublimate a white amorphous powder insoluble in boiling water, and with nitrate of silver a white precipitate insoluble in cold and sparingly in boiling water, from which the compound is obtained in crystals on cooling. Its most characteristic reactions, however, are those it gives with the ferro- and ferri-cyanides of potassium. If a few drops of the ferrocyanide of potassium be added to a not too dilute solution of the dipyridine hydrochlorate, a pale precipitate makes its appearance, which rapidly changes to a dirty indigo colour, increasing at the same time in quantity. If the proper concentration is hit, the precipitate consists entirely of very minute needle-shaped crystals having a dark indigo colour. They dissolve in boiling water, forming a very deep and rather dull purple solution, and are again deposited on cooling; but if the boiling be continued for some time, the compound appears to undergo some change, for the fluid retains its red colour at ordinary temperatures, though a great part of the substance is still deposited in crystals. A saturated cold solution of dipyridine in water gives no precipitate with ferrocyanide of potassium, but on the addition to the mixture of a drop or two of hydrochloric acid the dark precipitate instantly makes its appearance, and is deposited in small crystals. The precipitate is readily soluble in excess of hydrochloric acid. When ferricyanide of potassium is added to dipyridine hydrochlorate no immediate effect is observed, but on standing, the interior of the test-tube becomes lined with minute prisms of sulphur yellow colour and high lustre. If the solution be boiled, it acquires a dark colour, and partial decomposition takes place.

Dipyridine carefully dried in the water bath was found on analysis to give these results:—

I. 0·2802 gramme dipyridine gave
0·7782 ... carbonic acid, and
0·1497 ... water.

II. 0·3000 gramme dipyridine gave
0·8367 ... carbonic acid, and
0·1575 ... water.

	Experiment.		Calculation.		
	I.	II.			
Carbon,	75·74	76·07	75·94	C_5	60
Hydrogen,	5·94	5·83	6·33	H_5	5
Nitrogen,	...	...	17·73	N	14
			100·00		79

These numbers lead to the formula C_5H_5N, which is that of pyridine itself.

The platinum compound of the base, which is thrown down as a yellow crystalline powder, gave the following results:—

I. 0·3345 gramme gave
0·1175 ... platinum.

II. 0·2007 gramme gave
0·0685 ... platinum.

III. 0·4335 gramme gave
0·1473 ... platinum.

IV. 0·4052 gramme gave
0·3087 ... carbonic acid, and
0·0905 ... water.

	Experiment.			Calculation.		
	I.	II.	III.			
Carbon,	...	...	20·78	21·02	C_{10}	120
Hydrogen,	...	...	2·48	2·10	H_{12}	12
Nitrogen,	...	...	...	4·90	N_2	28
Chlorine,	...	...	...	37·30	Cl_6	213
Platinum,	34·11	34·12	34·03	34·68	Pt	197·4
				100·00		570·4

This agrees with the formula $(C_5H_5NHCl)_2PtCl_4$, which is identical with that of the pyridine salt. In order to fix the true constitution of the base, it was necessary to determine its vapour density, and as its boiling point was beyond the range of the mercurial thermometer, it was necessary to use a bath of metallic lead and an air thermometer. The air thermometer was a bulb of the same size as that used for containing the vapour, and the details of the experiment were as follows:—

Weight of vapour bulb empty, .	22·2530
... ... full of vapour,	22·4806
Temperature at weighing, . .	13°5
Barometer, ...	753 mm.
Volume of bulb, ...	127·1 C.C.
Residual air, ...	0·2 C.C.

Hence

Weight of bulb and vapour, . .	22·4806
... air displaced, . .	0·1552
Weight of bulb and vapour in vacuo,	22·6358
... bulb, ...	22·5230
	0·3828
Residual air, ...	0·0003
Weight of vapour, ...	0·3825
Weight of air thermometer, . .	264 grains.
... partly full of mercury, .	14391
Barometer 753 – 67·5 . . .	685·5 mm.
Temperature, ...	17°·5 C.
Air thermometer full of mercury, .	26930

Hence

Weight of mercury partially filling bulb,	14127
... ... completely, .	26666

and

$$26666 - 14127 = 12539.$$

Now, we have here in the formula

$$W(1 + kt) = w.\frac{h}{H} \times \frac{1 + at}{1 + at'}.$$

$W = 26666$

$w = 12539$

$H = 753$

$h = 685{\cdot}5$

$a = 0{\cdot}00366$

$k = 0{\cdot}00003$

$t' = 17^{\circ}{\cdot}5$

whence

$$t = 414^{\circ}{\cdot}4 \text{ Cent.}$$

Now

$$127{\cdot}1 - 0{\cdot}5 = 126{\cdot}5,^{*}$$

and

$$\frac{126{\cdot}5}{1 + 414^{\circ}{\cdot}4 \times 0{\cdot}00366} \times \frac{753}{760} \times \frac{1{\cdot}29366}{1000} = 0{\cdot}0646\,,$$

and

$$\frac{0{\cdot}3825}{0{\cdot}0646} = 5{\cdot}92\,.$$

The formula $C_{10}H_{10}N_2$ requires 5·46. This result is as close as could be expected under the circumstances, and proves that the base must be formed by the combination of two molecules of pyridine, and hence the name of dipyridine which I have applied to it.

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* 0·5 is the volume of the residual air at 414°.

Salts of Dipyridine.—Though dipyridine is not a very powerful base, it gives a number of salts, most of which crystallise well, though some of them are not easily obtained of definite composition.

Hydrochlorate of Dipyridine.—This salt is best obtained by adding a slight excess of hydrochloric acid to the aqueous solution of the base and evaporating to crystallisation; the crystals, after being pressed and recrystallised from water, are sufficiently pure for analysis. They are flat needles readily soluble in water, especially when hot—insoluble in ether. The salt is very apt to retain hydrochloric acid, and it is advisable to heat it to 130° for analysis.

0·6104 gramme of the hydrochlorate gave
0·7633 .. iodide of silver.

	Experiment.	Calculation.		
Carbon,	...	51·94	C_{10}	120
Hydrogen,	...	5·19	H_{12}	12
Nitrogen,	...	12·13	N_2	28
Chlorine,	30·93	30·74	Cl_2	71
		100·00		231

Sulphate of Dipyridine.—Dipyridine is added in slight excess to dilute sulphuric acid, and this fluid is evaporated nearly to dryness; on cooling, crystals of the sulphate are deposited; they are washed with alcohol, in which they are scarcely soluble, and again crystallised from water. It is thus obtained in needle-shaped crystals, which deliquesce in moist air. One determination of sulphuric acid was made of a specimen of this salt dried in vacuo over sulphuric acid, which gave 26·85 per cent. of SO_3. This would correspond with a salt containing two molecules of water of crystallisation with the formula $C_{10}H_{10}N_2H_2SO_4 + 2H_2O$, which requires 27·39 per cent.

Nitrate of Dipyridine is obtained by adding a slight excess of nitric acid to solution of dipyridine, evaporating on the water-bath and recrystallising. It forms pale yellow needles of moderate solubility in water. When exposed for some time to a temperature of 100°, it acquires an orange colour, but is not sensibly decomposed. A combustion of this salt gave

I. 0·5162 gramme of the nitrate gave
0·7915 ... of carbonic acid, and
0·1820 ... of water.

	Experiment.	Calculation.		
Carbon,	41·81	42·25	C_{10}	120
Hydrogen,	3·92	3·92	H_{12}	12
Nitrogen,	...	20·38	N_4	56
Oxygen,	...	33·45	O_6	96
		100·00		284

Double Salt with Chloride of Zinc.—This substance is prepared by adding to dipyridine a mixture of hydrochloric acid and zinc chloride, and allowing it to stand for some time, when small prismatic crystals are deposited. Their formation is materially assisted by the addition of alcohol and ether. The compound may even be obtained from the crude product, in which the dipyridine is mixed with the oily base distilling along with it. For this purpose zinc chloride, along with hydrochloric acid and a considerable excess of alcohol and ether, must be added to the crude product, when, especially on stirring, the zinc salt deposits, and can be purified by crystallisation from water. This process can even be employed for separating the two bases. The salt is in long white needles, soluble in eight times their weight of water, less so in alcohol, and quite insoluble in ether. On the addition of potassic hydrate in excess it gives the pure base in minute crystals. A chlorine determination gave

$$\begin{cases} 0{\cdot}3660 \text{ gramme zinc salt gave} \\ 0{\cdot}5755 \quad \ldots \quad \text{dichloride of zinc.} \end{cases}$$

	Experiment.	Calculation.		
Carbon, . .	...	32·70	C_{10}	120
Hydrogen, . .	...	3·27	H_{12}	12
Nitrogen, . .	...	7·62	N_2	28
Chlorine, . .	38·90	38·68	Cl_4	142
Zinc, . . .	...	17·73	Zn	65
		100·00		367

Double Salt with Nitrate of Silver.—This salt is best obtained by mixing hot solutions of dipyridine hydrochlorate, and silver nitrate, the latter being in excess, and at once filtering off the precipitated silver chloride. On cooling, the salt is deposited in brilliant needles, of sparing solubility in water. This compound could not be obtained of constant composition, but one specimen gave 35·09 per cent. of silver, while the formula $C_{10}H_{10}N_2(HNO_3)_2(AgNO_3)_2$ requires 34·61.

Platino-chloride of Dipyridine.—The analysis of this salt has been already given. It is obtained as a crystalline yellow powder of very sparing solubility.

Palladio-chloride of Dipyridine is obtained as an orange precipitate on mixing the hydrochlorates.

DERIVATIVES OF DIPYRIDINE.

Diethylo-Dipyridine.—The compounds of this base were obtained in the usual manner. The ethyl-iodide is easily prepared by exposing dry dipyridine with iodide of ethyl to the temperature of 100° in hermetically sealed tubes. The action is complete in half an hour. It is obtained in acicular crystals, which are brilliant and perfectly colourless if they have not been exposed to the air. They are very soluble in water, much less so in alcohol and in ether.

I. 0·3997 gramme ethyl-iodide gave
0·5195 ... carbonic acid, and
0·1530 ... water.

II. 0·3512 gramme gave
0·3513 ... iodide of silver.

	Experiment.	Calculation.		
Carbon,	35·44	35·74	C_{14}	168
Hydrogen,	4·25	4·26	H_{20}	20
Nitrogen,	...	5·96	N_2	28
Iodine,	54·05	54·04	I_2	254
		100·00		470

Corresponding with the formula $C_{10}H_{20}N_2(C_2H_5I)_2$. Heated with silver chloride and water this salt was converted into the chloride, which, on the addition of platinic-chloride gave the platino-chloride in very sparingly soluble small red needles.

0·2780 gramme platino-chloride gave
0·0685 ... platinum.

This corresponds with 31·12 per cent., and the formula $C_{10}H_{10}N_2(C_2H_5Cl)_2PtCl_4$ requires 30·44.

The base itself, when separated from the iodide by silver oxide, forms a highly alkaline solution, having generally a red or purple colour, which, on evaporation, leaves a dark-coloured uncrystallised residue. It obviously belongs to the class of ammonium bases, but I have not pursued its investigation further.

Dibromo-Dipyridine.—This base is thrown down when bromine is added to a solution of dipyridine hydrochlorate, or hydrobromate, as a white powder, insoluble in water, sparingly soluble in cold, more so in hot alcohol, from which it is deposited in flattened needles on cooling. If too much bromine has been used in its preparation, these crystals are pink. Its basic properties are extremely feeble, and it is a somewhat unstable compound; for, on boiling with water, or with hydrochloric acid, the original base appears to be more or less completely regenerated. An analysis, in which the hydrogen was lost, gave

I. 0·3560 gramme dibromo-dipyridine gave
0·4902 ... carbonic acid.

II. 0·3420 gramme gave
0·4078 .. silver iodide.

	Experiment.	Calculation.		
Carbon,	37·56	37·97	C_{10}	120
Hydrogen,	...	2·53	H_8	8
Nitrogen,	...	8·87	N_2	28
Bromine,	50·74	50·87	Br_2	160
		100·00		316

Its platinum compound could not be obtained in a state fitted for analysis. On boiling with hydrochloric acid, and adding platinic-chloride, a yellow precipitate was obtained, which, in one experiment, gave 32·39 per cent. of platinum; and in another, in which the boiling was continued longer, 33·53 per cent. was obtained. Dibromo-dipyridine requires 30·94, and dipyridine itself 34·68; so that there can be little doubt that the latter has been regenerated.

OILY BASE.

The oily base from which the dipyridine was deposited in crystals, has been as yet but imperfectly examined. It was purified by redistillation and cooling, by which it yielded a small additional quantity of dipyridine, and this was repeated as long as it gave crystals. The base so obtained is a rather thick, pale-yellow oil, heavier than water, having a peculiar heavy smell, quite distinct from that of pyridine. It is insoluble in water, but dissolves with great ease in alcohol and ether. It boils at a high temperature, and if distilled rapidly, it undergoes partial decomposition, yielding a small quantity of what appears to be a mixture of several bases with pungent smell, and sparingly soluble in water. If, however, the distillation be carried on very cautiously at a temperature below its boiling point, it passes over unchanged. It dissolves in acids and forms salts, most of which, however, are uncrystallisable, and dry up into gummy masses. It was prepared for analysis by drying over calcic-chloride, distilling and separating the first part of the distillate which might retain moisture.

I. { 0·3420 gramme of the base gave
0·9605 ... carbonic acid.
0·2135 ... water.

II. { 0·3662 gramme of the base gave
1·0140 ... carbonic acid.
0·2300 ... water.

	Experiment.		Calculation.		
	I.	II.			
Carbon, . .	76·59	75·51	75·94	C_5	60
Hydrogen, .	6·94	6·98	6·33	H_5	5
Nitrogen, . .	...	...	17·73	N	14
			100·00		79

These results, it will be seen, correspond with those given by pyridine and dipyridine, and they are confirmed by the analysis of a platinum salt, prepared in the usual way.

{ 0·2991 gramme of the platinum salt gave
0·1015 ... platinum.

This corresponds to 33·94, and calculation for the formula $(C_5H_5NHCl)_2PtCl_4$ requires 34·68. It is obvious, therefore, that this is another polymer of the original pyridine ; but, unfortunately, there is no means by which its molecular constitution can be determined. It is impossible to determine its vapour density, because it undergoes partial decomposition at its boiling point ; and as its salts do not crystallise, and probably, like the platinum compound, all correspond with those of the original pyridine, there is no prospect of satisfactory conclusions being drawn from them. In the absence of experimental evidence, any assumption may be made regarding the constitution of this base, and at first sight the most reasonable view of the matter is to suppose it to be the product of a further polymerisation, and to be formed by the combination of three or four molecules of the original pyridine. Its boiling point, which is certainly lower than that of dipyridine, however, appears to militate against this view ; and taking its properties and those of its compounds into consideration, I am inclined to believe it to be another dipyridine, and an example of those cases of physical isomerism of which so many are now known. As there was no means of ascertaining the constitution of this base, and the properties of its compounds were not encouraging, I have not pursued their investigation further.

Light Bases.

It has been stated at the commencement of this paper that when dipyridine was prepared by the first of the processes there described, that a light basic oil was obtained at the beginning of the rectification of the crude product. This oil, which is insoluble in water, was collected, dried, and rectified when it was found to consist of several bases. The distillate was collected in several fractions, which were analysed, but the quantity was far too small to admit of any systematic attempt to separate them. The results, both of the combustion of the bases themselves and of the platinum determinations in their platinum compounds, seem to show that they are a class of bases isologous with the pyridine series. I give here the results of these analyses :—

I.	0·3025	gramme of base boiling between 225° and 240° F. gave
	0·8375	... ... carbonic acid, and
	0·2302	... ... water.
II.	0·4120	gramme of base boiling between 270° and 290° F. gave
	1·1385	... ... carbonic acid, and
	0·3230	... ... water.
III.	0·2711	gramme of base boiling about 291° F. gave
	0·7572	... ... carbonic acid, and
	0·1984	... ... water.
IV.	0·2708	gramme of base boiling between 287° and 291° F. gave
	0·7371	... ... carbonic acid, and
	0·2092	... ... water.
V.	0·2355	gramme of base boiling between 291° and 360° F. gave
	0·6690	... ... carbonic acid,
	0·1795	... ... water.

	I.	II.	III.	IV.	V.
Carbon,	75·51	75·36	76·18	74·23	77·47
Hydrogen,	8·46	8·71	8·13	8·58	8·47
Nitrogen,	16·03	15·93	15·69	17·19	14·06
	100·00	100·00	100·00	100·00	100·00

It must be distinctly understood that all the substances analysed were obviously mixtures, and the degrees above given do not denote true boiling points, but only that the fractions they represent were collected at these points. It will be observed that all these analyses are characterised by yielding a percentage of hydrogen far above that contained in pyridine, or any of its homologues. To render this more obvious, I place here the calculated numbers for pyridine and picoline, along with those required by bases containing two atoms of hydrogen more than these compounds:—

	C_5H_5N	C_6H_7N	C_5H_7N	C_6H_9N
Carbon,	75·94	77·42	74·04	75·75
Hydrogen,	6·33	7·53	8·64	9·47
Nitrogen,	17·73	15·05	17·32	14·78
	100·00	100·00	100·00	100·00

The experimental results above given are manifestly incompatible with the first two of these formula, but would agree tolerably with a mixture of substances containing more hydrogen, and this is further confirmed by several platinum determinations in platinum compounds obtained from these substances. These results appear to indicate the existence of a series of bases having the general formula $C_nH_{2n-3}N$. The further investigation of these substances would be of interest, but as they are obtained only in minute quantity, and are clearly secondary products of the action of sodium on pyridine, it is scarcely possible to obtain them in sufficient quantity for this purpose.

At the beginning of this paper, mention has been made of a black or dark-gray substance, obtained in the second process for preparing dipyridine. This compound was only obtained towards the close of the investigation, and I have not had time to examine its properties and relations minutely. It is a black amorphous powder, quite insoluble in water. When exposed to the air, it is rapidly converted into a mass of crystals of pure dipyridine. I was at first disposed to consider this substance to be a sodium compound of dipyridine, but I soon found that this was not the case, and that its properties more nearly corresponded with a hydrogen compound of that base; and it seems probable that its formula will turn out to be $C_{10}H_{12}N_2$, in which case it would be related to dipyridine in the same manner as ammonium is to ammonia. Should this view be correct, it seems probable that, on the addition of hydrochloric acid, it should, like a metal, evolve hydrogen. An experiment was made to ascertain whether this occurred by introducing a quantity of the compound into a jar over mercury, and bringing hydrochloric acid in contact with it; but the anticipated result was not obtained,—no hydrogen was evolved, but a brownish precipitate appeared in the fluid, and the gray powder at once disappeared. I have been unable to pursue this subject further, but propose to return to it on a future occasion.

Numerous experiments have been made, in the hope of throwing light on the nature of the chemical changes occurring during the first process of preparing dipyridine; but it is obviously of a very complex kind, and some of the products must be the result of secondary decompositions. My impression is, that sodio-dipyridine must be first formed, probably $C_{10}H_8Na_2N_2$. In this case hydrogen must be given off during the action, and this is actually the case, as was established by direct experiment; but the quantity evolved is trifling compared with that of the sodium consumed, so that if the action takes place in this way, a large part of the hydrogen must be converted into some other compounds within the mixture itself. The light bases already mentioned might account for this, if it were not that they are produced in very small quantity. Altogether I am inclined to think that, in the first process, a number of secondary reactions take place, which greatly complicate matters, and that it is through some modification of the second by which an explanation will most probably be obtained. I am still engaged with the subject, and have already nearly perfected a process by which some of the products can be obtained with greater certainty and in larger quantity than by either of those described in this paper, and which I hope will enable me to subject the constitution and relations of these curious compounds to a more minute examination.

VI.—*On Vortex Motion.* By Sir W. Thomson.

(Read 29th April 1867.)

(§§ 1–59 recast and augmented 28th August to 12th November 1868.)

1. The mathematical work of the present paper has been performed to illustrate the hypothesis, that space is continuously occupied by an incompressible frictionless liquid acted on by no force, and that material phenomena of every kind depend solely on motions created in this liquid. But I take, in the first place, as subject of investigation, a finite mass of incompressible frictionless* fluid completely enclosed in a rigid fixed boundary.

2. The containing vessel may be either *simply* or *multiply continuous.*† And I shall frequently consider solids surrounded by the liquid, which also may be either simply or multiply continuous. It will not be necessary to exclude the supposition that any such solid may touch the outer boundary over some finite area, in which case it is *not* surrounded by the liquid; but each such solid, whether surrounded by the liquid or not, and whether moveable or fixed, must be considered as a part of the whole boundary of the liquid.

3. Let the whole fluid be given at rest, and let no force, except pressure from the containing vessel, or from the surfaces of solids immersed in it, ever act on any part of it. Let there be any number of solids, perfectly incompressible, and of the same density as the fluid; but either perfectly rigid, or more or less flexible, with perfect or imperfect elasticity. Some of these may at times be supposed to lose rigidity, and become perfectly liquid; and portions of the liquid may be supposed to acquire rigidity, and thus to constitute solids. Let the solids act on one another with any forces, pressures, frictions, or mutual distant actions, subject only to the law of "action and reaction." Let motions originate among them and in the liquid, either by the natural mutual actions of the solids or by the arbitrary application of forces to them during some limited time. It is of no consequence to us whether these forces have reactions on matter outside the containing vessel, so that they might be called "natural forces" in the present state of science (which admits action and reaction at a distance); or are applied arbitrarily by supernatural action without reaction. To avoid circumlocution,

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* A frictionless fluid is defined as a mass continuously occupying space, whose contiguous portions press on one another everywhere exactly in the direction perpendicular to the surface separating them.

† Helmholtz—*Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen:* Crelle (1858); translated by Tait in Phil. Mag. 1867, i. Riemann—*Lehrsätze aus der Analysis situs, &c.* Crelle (1857). See also § 58, below.

and, at the same time, to conform to a common usage, we shall call them *impressed forces.*

4. From the homogeneousness as to density of the contents of the fixed bounding vessel, it follows that the centre of inertia of the whole system of liquid and solids immersed in it remains at rest; in other words, the integral momentum of the motion is zero. Hence (THOMSON and TAIT'S "Natural Philosophy," § 297) the time integral of the sum of the components of *pressure on the containing vessel*, parallel to any fixed line, is equal to the time-integral of the sum of the components of *impressed forces* parallel to the same line. This equality exists, of course, at each instant during the action of the impressed forces, and continues to exist for the constant values of their time integrals, after they have ceased. Thus, in the subsequent motion of the solids, and of the fluids compelled to yield to them, whatever pressure may come to act on the containing vessel, whether from the fluid or from some of the solids coming in contact with it, the components of this pressure, parallel to any fixed line, summed for every element of the inner surface of the vessel, must vanish for every interval of time during which no impressed forces act. If, for example, one of the solids strikes the containing vessel, there will be an impulsive pressure of the fluid over all the rest of the fixed containing surface, having the sum of its components parallel to any line, equal and contrary* to the corresponding component of the impulsive pressure of the solid on the part of this surface which it strikes [see § 8, and consider oblique impulse of an inner moving solid, on the fixed solid spherical boundary]. *But, after the impressed forces cease to act, and as long as the containing vessel is not touched by any of the solids, the integral amount of the component of fluid pressure on it, parallel to any line, vanishes.*

5. If now forces be applied to stop the whole motion of fluid and solids [as (§ 62) is done, if the solids are brought to rest by forces applied to themselves only], the time integrals of the sums of the components of these forces, parallel to any stated lines, *may or may not in general be equal and contrary* to the time integrals of the corresponding sums of components of the initiating impressed forces (§ 3). But we shall see (§§ 19, 21), that *if the containing vessel be infinitely large, and all of the moving solids be infinitely distant from it during the whole motion,* there must be not merely the equality in question between the time integrals of the components in contrary directions of the initiating and stopping impressed forces, but there must be (§ 21) *completely equilibrating opposition between the two systems.*

6. To avoid circumlocution, henceforth I shall use the unqualified term *impulse* to signify a system of impulsive forces, to be dealt with as if acting on a rigid body. Thus the most general impulse may be reduced to an impulsive force, and couple

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* I shall use the word *contrary* to designate merely directional opposition; and reserve the unqualified word *opposite*, to signify *contrary and in one line.*

in plane perpendicular to it, according to POINSOT; or to two impulsive forces in lines not meeting, according to his predecessors. Further, I shall designate by *the impulse of the motion at any instant*, in our present subject, the system of impulsive forces on the moveable solids which would generate it from rest; or any other system which would be equivalent to that one if the solids were all rigid and rigidly connected with one another, as, for instance, the POINSOT resultant impulsive force and minimum couple. The line of this resultant impulsive force will be called the *resultant axis of the motion*, and the moment of the minimum couple (whose plane is perpendicular to this line) will be called the *rotational moment* of the motion.

7. But, having thus defined the terms I intend to use, I must, to warn against errors that might be fallen into, remark that the momentum of the whole motions of solids and liquid is *not* equal to what I have defined as *the impulse*, but (§ 4) is equal to zero; being the force-resultant of "the impulse" and the impulsive pressure exerted on the liquid by the containing vessel during the generation of the motion: and that the moment of momentum of the whole motion round the centre of inertia of the contents of the vessel is *not* equal to the *rotational moment*, as I have defined it, but is equal to the moment of the couple constituted by "the impulse" and the impulsive pressure of the containing vessel on the liquid. It must be borne in mind that however large, and however distant all round from the moveable solids, the containing vessel may be, it exercises a finite influence on the momentum and moment of momentum of the whole motion within it. But if it is infinitely large, and infinitely distant all round from the solids, it does so by infinitely slow motion through an infinitely large mass of fluid, and exercises no finite influence on the finite motion of the solids or of the neighbouring fluid. This will be readily understood, if for an instant we suppose the rigid containing vessel to be not fixed, but quite free to move as a rigid body without mass. The momentum of the whole motion will then be not zero, but exactly equal to the force-resultant of the impulse on the solids; and the moment of momentum of the whole motion round the centre of inertia will be precisely equal to the resultant impulsive couple found by transposing the constituent impulsive forces to this point after the manner of POINSOT. But the finite motion of the immersed solids, and of the fluid in their neighbourhood which we shall call the *field of motion*, will not be altered by any finite difference, whether the containing vessel be held fixed or left free, provided it be infinitely distant from them all round. It is, therefore, essentially indifferent whether we keep it fixed or let it be free. The former supposition is more convenient in some respects, the latter in others; but it would be inconvenient to leave any ambiguity, and I shall adhere (§ 1) to the former in all that follows.

8. To further illustrate the impulse of the motion, and its resultant impulsive force and couple, according to the previous definitions, as distinguished from

the momentum, and the moment of momentum, of the whole contents of the vessel, let the vessel be spherical. Its impulsive pressure on the liquid will always be reducible to a single resultant in a line through its centre, which (§ 4) will be equal and contrary to the force-resultant of "the impulse;" and, therefore, with it will constitute in general a couple. The resultant, of this couple and the couple-resultant of the impulse, will be equal to the moment of momentum of the whole motion round the centre of the sphere (which is the centre of inertia). But if the vessel be infinitely large, and infinitely distant all round from the moveable solids, the moment of momentum of the whole motion is irrelevant; and what is essentially important, is the impulse and its force and couple-resultants, as defined above.

9. The following way of stating (§§ 10, 12), and proving (§§ 11—15), a fundamental proposition in fluid motion will be useful to us for the theory of the impulse, whether of the moveable solids we have hitherto considered or of vortices.

10. The moment of momentum of every spherical portion of a liquid mass in motion, relatively to the centre of the sphere, is always zero, if it is so at any one instant for every spherical portion of the same mass.

11. To prove this, it is first to be remarked, that the moment of momentum of that part of the liquid which at any instant occupies a certain fixed spherical space can experience no change, at that instant (or its *rate* of change vanishes at that instant), because the fluid pressure on it (§ 1), being perpendicular to its surface, is everywhere precisely towards its centre. Hence, if the moment of momentum of the matter in the fixed spherical space varies, it must be by the moment of momentum of the matter which enters it not balancing exactly that of the matter which leaves it. We shall see later (§§ 20, 17, 18) that this balancing is vitiated by the entry of either a moving solid, or of some of the liquid, if any there is, of which spherical portions possess moment of momentum, into the fixed spherical space; but it is perfect under the condition of § 10, as will be proved in § 15.

12. First, I shall prove the following purely mathematical lemmas; using the ordinary notation u, v, w for the components of fluid velocity at any point (x, y, z).

Lemma (1.) The condition (last clause) of § 10 requires that $u\,dx + v\,dy + w\,dz$ be a complete differential,* at whatever instant and through whatever part of the fluid the condition holds.

Lemma (2.) If $u\,dx + v\,dy + w\,dz$ be a complete differential of a single valued function of x, y, z, through any finite space of the fluid, at any instant, the condition of § 10 holds through that space at that instant.

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* This proposition was, I believe, first proved by STOKES in his paper "On the Friction of Fluids in Motion, and the Equilibrium and Motion of Elastic Solids."—"Cambridge Philosophical Transactions," 14th April 1845.

13. The following is STOKES' proof of Lemma (1):—First, for any motion whatever, whether subject to the condition of § 10 or not, let L be the component moment of momentum round OX of an infinitesimal sphere with its centre at O. Denoting by $\iiint$ integration through this space, we have

$$\mathrm{L} = \iiint (wy - vz)\, dx\, dy\, dz \quad . \quad . \quad . \quad (1).$$

Now let $\left(\frac{dw}{dx}\right)_0$, $\left(\frac{dw}{dy}\right)_0$, &c. denote the values at O of the differential coefficients. We have, by MACLAURIN's theorem,

$$w = x\left(\frac{dw}{dx}\right)_0 + y\left(\frac{dw}{dy}\right)_0 + z\left(\frac{dw}{dz}\right)_0 ,$$

and so for v. Hence, remembering that $\left(\frac{dw}{dx}\right)_0$, &c. are constants for the space through which the integration is performed, we have

$$\iiint dx\, dy\, dz\, wy = \left(\frac{dw}{dx}\right)_0 \iiint xy\, dx\, dy\, dz + \left(\frac{dw}{dy}\right)_0 \iiint y^2 dx\, dy\, dz + \left(\frac{dw}{dz}\right)_0 \iiint zy\, dx\, dy\, dz .$$

The first and third of the triple integrals vanish, because every diameter of a homogeneous sphere is a principal axis; and if A denote moment of momentum of the spherical volume round its centre, we have for the second

$$\iiint y^2\, dx\, dy\, dz = \tfrac{1}{2}\mathrm{A} .$$

Dealing similarly with vz in the expression for L, we find

$$\mathrm{L} = \tfrac{1}{2}\mathrm{A}\left[\left(\frac{dw}{dy}\right)_0 - \left(\frac{dv}{dz}\right)_0\right] \quad . \quad . \quad . \quad (2).$$

But L must be zero according to the condition of § 10; and, therefore, as the centre of the infinitesimal sphere now considered may be taken at any point of space through which this condition holds at any instant, we must have, throughout that space,

$$\text{and similarly} \quad \left.\begin{aligned} \frac{dw}{dy} - \frac{dv}{dz} &= 0 \\ \frac{du}{dz} - \frac{dw}{dx} &= 0 \\ \frac{dv}{dx} - \frac{du}{dy} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3);$$

which proves Lemma (1.)

14. To prove Lemma (2.), let

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{dx}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

and let L denote the component moment of momentum round OX, through any spherical space with O in centre. We have [(1) of § 13],

$$\mathrm{L} = \iiint dx\, dy\, dz\, (wy - vz) \quad . \quad . \quad . \quad . \quad . \quad (5),$$

$\iint$ denoting integration throughout this space (not now infinitesimal). But by (4)

$$yw - vz = \left(y\frac{d}{dz} - z\frac{d}{dy}\right)\varphi = \frac{d\varphi}{d\psi} \quad . \quad (6);$$

if $\frac{d}{d\psi}$ denote differentiation with reference to ψ, in the system of co-ordinate x, ρ, ψ, such that

$$y = \varrho \cos \psi\,,\ z = \varrho \sin \psi \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Hence, transforming (5) to this system of co-ordinates, we have

$$\mathrm{L} = \iiint dx\, d\varrho\, \varrho d\psi\, \frac{d\varphi}{d\psi} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Now, as the whole space is spherical, with the origin of co-ordinates in its centre, we may divide it into infinitesimal circular rings with OX for axis, having each for normal section an infinitesimal rectangle with dx and $d\rho$ for sides. Integrating first through one of these rings, we have

$$dx\, d\varrho\, \varrho \int_0^{2\pi} \frac{d\varphi}{d\psi}\, d\psi\,,$$

which vanishes, because φ is a single-valued function of the co-ordinates. Hence $\mathrm{L} = 0$, which proves Lemma (2.).

15. Returning now to the dynamical proposition, stated at the conclusion of § 11; for the promised proof, let R denote the radial component velocity of the fluid across any element, $d\sigma$, of the spherical surface, situated at (x, y, z); and let u, v, w be the three components of the resultant velocity at this point; so that

$$\mathrm{R} = u\frac{x}{r} + v\frac{y}{r} + w\,\frac{z}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

The volume of fluid leaving the hollow spherical space across $d\sigma$ in an infinitesimal time, dt is $\mathrm{R}d\sigma\,.\,dt$, and the moment of momentum of this moving mass round the centre has, for component round OX,

$$(wy - vz)\, \mathrm{R}d\sigma\, dt.$$

Hence, if L denote the component of the moment of momentum of the whole, mass within the spherical surface at any instant, t, we have (§ 11),

$$\frac{d\mathrm{L}}{dt} = \iint (wy - vz)\, \mathrm{R}\, d\sigma, \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Now, using Lemma (1.) of § 12, and the notation of § 14, we have

$$wy - vz = \frac{d\varphi}{d\psi},$$

and, by (9),

$$\mathrm{R} = \frac{d\varphi}{dr}$$

where $\frac{d}{dr}$ denotes rate of variation per unit length perpendicular to the spherical surface, that is differentiation with reference to r, the other two co-ordinates being directional relatively to the centre. Hence, using ordinary polar co-ordinates, r, θ, ψ, we have

$$\frac{d\mathrm{L}}{dt} = r^2 \iint \frac{d\varphi}{dr}\frac{d\varphi}{d\psi} \sin\theta \, d\theta \, d\psi \quad . \quad . \quad . \quad . \quad (11).$$

But the "equation of continuity" for an incompressible liquid (being

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0),$$

gives* $\nabla^2 \phi = 0$, for every point within the spherical space; and therefore [THOMSON & TAIT, App. B]

$$\varphi = \mathrm{S}_0 + \mathrm{S}_1 r + \mathrm{S}_2 r^2 + \&c. \quad . \quad . \quad . \quad . \quad (12).$$

a converging series, where S_0 denotes a constant, and S_1, S_2, &c., surface harmonics of the orders indicated.

Hence

$$\mathrm{R} = \frac{d\varphi}{dr} = \mathrm{S}_1 + 2r\,\mathrm{S}_2 + 3\,r^2\mathrm{S}_3 + \&c. \quad . \quad . \quad . \quad (13).$$

And it is clear from the synthesis of the most general surface harmonic, by zonal, sectional, and tesseral harmonics [THOMSON & TAIT, §781], that $\frac{d\mathrm{S}_i}{d\psi}$ is a surface harmonic of the same order as S_i:† from which [THOMSON & TAIT, App. B (16)], it follows that,

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[*] By ∇^2 we shall always understand $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$.

[†] This follows, of course, from the known analytical theorem that the operations ∇^2 and $\left(y\frac{d}{dz} - z\frac{d}{dy}\right)$ are commutative, which is proved thus:—

By differentiation we have

$$\frac{d^2\left(y\frac{d\varphi}{dz}\right)}{dy^2} = y\frac{d^2}{dy^2}\frac{d\varphi}{dz} + 2\frac{d}{dy}\frac{d\varphi}{dz};$$

and therefore, since $\frac{d}{dy}\frac{d\varphi}{dz} = \frac{d}{dz}\frac{d\varphi}{dy}$,

$$\nabla^2\left(y\frac{d\varphi}{dz} - z\frac{d\varphi}{dy}\right) = y\,\nabla^2\left(\frac{d\varphi}{dz}\right) - z\,\nabla^2\left(\frac{d\varphi}{dy}\right) = \left(y\frac{d}{dz} - z\frac{d}{dy}\right)\nabla^2\varphi$$

or

$$\nabla^2\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\varphi = \left(y\frac{d}{dz} - z\frac{d}{dy}\right)\nabla^2\varphi,$$

φ being any function whatever. Hence, if $\nabla^2\varphi = 0$ we have

$$\nabla^2\left(y\frac{d\varphi}{dz} - z\frac{d\varphi}{dy}\right) = 0\cdot$$

$$\iint \mathrm{S}_i \frac{d\mathrm{S}_{i'}}{d\psi} \sin\theta \, d\theta \, d\psi = 0 ,$$

except when $i' = i$. But this is true also when $i' = i$ because

$$\mathrm{S}_i \frac{d\mathrm{S}_i}{d\psi} = \tfrac{1}{2} \, d \, \frac{(\mathrm{S}_i^2)}{d\psi} ,$$

and therefore, as in § 14, the integration for ψ, from $\psi = 0$ to $\psi = 2\pi$ gives zero. Hence (11) gives

$$\frac{d\mathrm{L}}{dt} = 0 ,$$

This and § 11 establish § 10.

16. Lemma (1) of § 11, and § 10 now proved, show that in any motion whatever of an incompressible liquid, whether with solids immersed in it or not, $udx + vdy + wdz$ is always a complete differential through any portion of the fluid, for which it is a complete differential at any instant, to whatever shape and position of space this portion may be brought in the course of the motion. This is the ordinary statement of the fundamental proposition of fluid motion referred to in § 9, which was first discovered by LAGRANGE. (For another proof see § 60.) I have given the preceding demonstration, not so much because it is useful to look at mathematical structures from many different points of view, but (§ 19) because the dynamical considerations and the formulæ I have used are immediately available for establishing the theory of the impulse (§§ 3 . . . 8), of which a fundamental proposition was stated above (§ 5). To prove this proposition (in § 19) I now proceed.

17. Imagine any spherical surfaces to be described round a moveable solid or solids immersed in a liquid. The surrounding fluid can only press (§ 1) perpendicularly; and therefore when any motion is (§ 3) generated by impulsive forces applied to the solids, the moment round any diameter of the momentum of the matter within the spherical surface at the first instant, must be exactly equal to the moment of those impulsive forces round this line. And the moment round this line, of the momentum of the matter in the space between any two concentric spherical surfaces is zero, provided neither cuts any solid, and provided that, if there are any solids in this space, no impulse acts on them.

18. Hence, considering what we have defined as "the impulse of the motion," (§ 6), we see that its moment round any line is equal to the moment of momentum round the same line, of all the motion within any spherical surface having its centre in this line, and enclosing all the matter to which any constituent of the impulse is applied. This will still hold, though there are other solids not in the neighbourhood, and impulses are applied to them: provided the moments of momentum of those only which are within S are taken into account, and provided none of them is cut by S.

19. The statements of § 11, regarding fluid occupying at any instant a fixed spherical surface, are applicable without change to the fluids and solids occupying

the space bounded by S, because of our present condition, that no solid is cut by S. Hence every statement and formula of § 15, as far as equation (11), may be now applied to the matter within S; but instead of (12) we now have [THOMSON & TAIT, § 736], if we denote by T_1, T_2, &c., another set of surface spherical harmonics,

$$\left.\begin{aligned}\varphi = S_0 + S_1\, r \quad + S_2\, r^2 \quad + \&c.\\ + T_1\, r^{-2} + T_2\, r^{-3} + \&c.\end{aligned}\right\} \quad . \quad . \quad . \qquad (14).$$[*]

for all space between the greatest and smallest spherical surface concentric with S, and having no solids in it, because through all this space, § 16, and the equation of continuity prove that $\nabla^2 \phi = 0$. Hence, instead of (13), we now have

$$\left.\begin{aligned}R = \frac{d\varphi}{dr} = S_1 + 2r\, S_2 + 3r^2\, S_3, \&c.\\ - \frac{2}{r^3}\, T_1 - \frac{3}{r^4}\, T_2 - \frac{4}{r^5}\, T_3 + \&c.\end{aligned}\right\} \quad . \qquad (15).$$

Hence finally

$$\frac{dL}{dt} = \sum_{i=0}^{i=\infty} \iint \left[i\, S_i \frac{d\, T_i}{d\psi} - (i+1)\, T_i \frac{dS_i}{d\psi} \right] \sin\theta d\theta d\psi \quad . \qquad (16).$$

Now if, as assumed in § 5, neither any moveable solids, nor any part of the boundary exist within any finite distance of S all round; S_1, S_2, &c., must each be infinitely small: and therefore (16) gives $\frac{dL}{dt} = 0$. This proves the proposition asserted in § 5: because a system of forces cannot have zero moment round every line drawn through any finite portion of space, without having force-resultant and couple-resultant each equal to zero.

20. As the rigidity of the solids has not been taken into account, all or any of them may be liquefied (§ 3) without violating the demonstration of § 19. To save circumlocutions, I now define a *vortex* as a portion of fluid having any motion that it could not acquire by fluid pressure transmitted through itself from its boundary. Often, merely for brevity, I shall use the expression *a body* to denote either a solid or a vortex, or a group of solids or vortices.

21. The proposition thus proved may be now stated in terms of the definitions of § 6, which were not used in § 5, and so becomes simply this:—*The impulse of the motion of a solid or group of solids or vortices and the surrounding liquid remains constant as long as no disturbance is suffered from the influence of other solids or vortices, or of the containing vessel.*

This implies, of course (§ 6), that the magnitudes of the force-resultant and the rotational moment of the impulse remain constant, and the position of its axis invariable.

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[*] There is no term $\frac{T_0}{r}$, because this would give, in the integral of flow across the whole spherical surface, a finite amount of flow out of or into the space within, implying a generation or destruction of matter.

22. In POINSOT'S system of the statics of a rigid body we may pass from the resultant force and couple along and round the central axis to an equal resultant force along the parallel line through any point, and a greater couple the resultant of the former (or minimum) couple, and a couple in the plane of the two parallels, having its moment equal to the product of their distance into the resultant force. So we may pass from the force-resultant and rotational moment of the impulse along and round its axis, to an equal force-resultant and greater moment of impulse, by transferring the former to any point, Q, not in the axis (§ 6) of the motion. This greater moment is (§ 18) equal to the moment of momentum round the point Q, of the motion within any spherical surface described from Q as centre, which encloses all the vortices or moving solids.

23. Hence a group of solids or vortices which always keep within a spherical surface of finite radius, or a single body, moving in an infinite liquid, can have no permanent average motion of translation in any direction oblique to the direction of the force-resultant of the impulse, if there is a finite force-resultant. For the matter within a finite spherical surface enclosing the moving bodies or body, cannot have moment of momentum round the centre increasing to infinity.

24. But there may be motion of translation when the force-resultant of the impulse vanishes; and there will be, for example, in the case of a solid, shaped like the screw-propeller of a steamer, immersed in an infinite homogeneous liquid, and set in motion by a couple in a plane perpendicular to the axis of the screw.

25. And when the force-resultant of the impulse does *not* vanish, there may be no motion of translation, or there may even be translation in the direction opposite to it. Thus, for example, a rigid ring, with cyclic motion, established (§ 63) through it, will, if left at rest, remain at rest. And if at any time urged by an impulse in either direction in the line of the force-resultant of the impulse of the cyclic motion, it will commence and continue moving with an average motion of translation in that direction; a motion which will be uniform, and the same as if there were no cyclic motion, when the ring is symmetrical. If the translatory impulse is contrary to the cyclic impulse, but less in magnitude, the translation will be contrary to the whole force-resultant impulse.

If the translatory impulse is equal and opposite to the cyclic impulse, there will be translation with zero force-resultant impulse—another example of what is asserted in § 24. In this case, if the ring is plane and symmetrical, or of any other shape such that the cyclic motion (which, to fix ideas, we have supposed given first, with the ring at rest,) must have had only a force-resultant, and no rotational moment, we have a solid moving with a uniform motion of translation through a fluid, and both force and couple resultant of the whole impulse zero.

26. From §§ 21 and 4, we see that, however long the time of application of the impressed forces may be—provided only that, during the whole of it, the

solid or group of solids has been at an infinite distance from all other solids and from the containing vessel—the time integrals of the impressed forces parallel to three fixed axes, and of their moments round these lines, are equal to the six corresponding components of "the impulse" (§ 6).

27. If two groups, at first so far asunder as to exercise no sensible influence on one another, come together, the "impulse" of the whole system remains unchanged by any disturbance each may experience from the other, whether by impacts of the solids, or through motion and pressure of the surrounding fluid; and (§ 6) it is always reducible to the force-resultant along the central axis, and the minimum couple-resultant, of the two impulses reckoned as if applied to one rigid body. The same holds, of course, if one group separates into two so distant as to no longer exert any sensible influence on one another.

28. Hence whatever is lost of impulse perpendicular to a fixed plane, or of component rotational movement round a fixed line, by one group through collision with another, is gained by the other.

29. Two of the moveable solids, or two groups, will be said to be *in collision* when, having been so far asunder as not to disturb one another's motions sensibly, they are so near as to do so. This disturbance will generally be supposed to be through fluid pressure only, but impacts of solids on solids may take place during a collision.

30. We are now prepared to investigate (§§ 30, 31, 32) the influence of a fixed solid on the impulse of a moveable solid, or of a vortex, or of a group of solids or vortices, passing near it, thus—If during such collisions or separations as are considered in §§ 27, 28, forces are impressed on any one or more of the solids, their alteration of the whole impulse is (§ 26) to be reckoned by adding to each of its rectangular components the time integral of the corresponding component of these impressed forces. Now, let us suppose such forces to be impressed on any one of the moveable solids as shall keep it at rest. These forces are zero as long as no moving solid is within a finite distance. But if a moving solid or vortex, or group of solids or vortices, passes near the fixed solid, the change of pressure due to the motion of the fluid will tend to move it, and the impression of force on it becomes necessary to keep it fixed. Let $d\sigma$ be an element of its surface; (x, y, z), the co-ordinates of the centre of this element; α, β, γ the inclinations of the normal at (x, y, z) to the three rectangular axes; and p the fluid pressure at time t, and point (x, y, z). The six components of force and couple required to hold the body fixed at time t, are

$$\left.\begin{array}{l} \iint d\sigma \,.\, \cos\alpha \,.\, p,\ \iint d\sigma \,.\, \cos\beta \,.\, p,\ \iint d\sigma \,.\, \cos\gamma \,.\, p\,; \\ \iint d\sigma\,(y\cos\gamma - z\cos\beta)p,\ \iint d\sigma(z\cos\alpha - x\cos\gamma)p,\ \iint d\sigma(x\cos\beta - y\cos x)p, \end{array}\right\} \quad (1).$$

If in these expressions we substitute

$$\int p\,dt \quad . \quad . \quad . \quad . \quad . \quad (2)\text{:}$$

in place of p ($\int dt$ denoting a time integral from any era of reckoning before the disturbance became sensible, up to time t, which may be any instant during the collision, or after it is finished), we have the changes in the corresponding components of the impulse up to time t, provided there has been no impact of moveable solid on the fixed solid.

31. Let now the "velocity potential" (as we shall call it, in conformity with a German usage which has been adopted by HELMHOLTZ,) be denoted by ϕ; that is (§ 16), let ϕ be such a function of (x, y, z, t) that

$$u = \frac{d\varphi}{dx}, \quad v = \frac{d\varphi}{dy}, \quad w = \frac{d\varphi}{dz} \quad . \quad . \quad . \quad (3).$$

and let $\dot{\phi}$ $\left(\text{or } \frac{d\varphi}{dt}\right)$ denote its rate of variation per unit of time at any instant t, for the point (x, y, z) regarded as fixed.

Also, let q denote the resultant fluid velocity, so that

$$q^2 = u^2 + v^2 + w^2 = \frac{d\varphi^2}{dx^2} + \frac{d\varphi^2}{dy^2} + \frac{d\varphi^2}{dz^2} \quad . \quad . \quad (4).$$

The ordinary hydro-dynamical formula gives

$$p = \Pi - \dot{\phi} - \tfrac{1}{2} q^2 \quad . \quad . \quad . \quad . \quad (5);$$

where Π denotes the constant pressure in all sensibly quiescent parts of the fluid.

32. The constant term Π disappears from p in each of the integrals (1) of § 30, because a solid is equilibrated by equal pressure around. And in the time integral (2), we have

$$\int \dot{\varphi}\, dt = \varphi \quad . \quad . \quad . \quad . \quad . \quad (6);$$

and therefere if (XYZ) (LMN) denote the changes in the force-and couple-components of the impulse produced by the collision up to time t, we have

$$\left.\begin{aligned} X &= -\iint d\sigma \cos\alpha\, (\varphi + \tfrac{1}{2}\int q^2\, dt),\ Y = \&c., Z = \&c., \\ L &= -\iint d\sigma\, (y \cos\gamma - z \cos\beta)\, (\varphi + \tfrac{1}{2}\int q^2\, dt), M = \&c., N = \&c., \end{aligned}\right\} \quad . \quad . \quad . \quad (7).$$

But because the fluid is quiescent in the neighbourhood of the fixed body when the moving body or group of bodies is infinitely distant from it; it follows that before the commencement and after the end of the collision we have $\phi = 0$ at every point of the surface of the fixed body. Hence, for every value of t representing a time after the completion of the collision, the preceding expressions become

$$\left.\begin{aligned} X &= -\tfrac{1}{2}\iint d\sigma \cos\alpha \int q^2\, dt,\ Y = \&c., Z = \&c., \\ L &= -\tfrac{1}{2}\iint d\sigma (y \cos\gamma - z \cos\beta) \int q^2\, dt, M = \&c., N = \&c., \end{aligned}\right\} \quad . \quad . \quad (8);$$

which express that *the integral change of impulse experienced by a body or group of bodies, in passing beside a fixed body without striking it, may be regarded as a* *system of impulsive attractions towards the latter, everywhere in the direction of the normal, and amounting to* $\frac{1}{2}\int q^2 dt$ *per unit of area.* But it must not be forgotten that the term ϕ in the expression [§ 31 (5)] for p produces, as shown in § 30 (1), an influence *during the collision*, the integral effect of which only disappears from the expression [§ 32 (7)] for the impulse *after the collision is completed;* that is (§ 29) after the moving system has passed away so far as to leave no sensible fluid motion in the neighbourhood of the fixed body.

33. Hence, and from § 23, we see that when there is no impact of moving solid against the fixed body, and when the moving solid or group of solids passes altogether on one side of the fixed body, the direction of the translation will be deflected, as if there were, on the whole, an *attraction towards* the fixed body, or a *repulsion from it*, according as (§ 25) the translation is in the direction of the impulse or opposite to it. For, in each case, the impulse is altered by the introduction of an impulse *towards* the fixed body upon the moving body or bodies as they pass it; and (§ 23) the translation before and after the collision is always along the line of the impulse, and is altered in direction accordingly. This will be easily understood from the diagrams, where, in each case B represents the fixed body, the dotted line ITT′I′, and arrow-heads I I′, the directions of the force-resultant of the impulse at successive times, and the full arrow-heads T T′, the directions of the translation.

Fig. 1

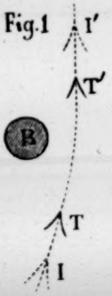
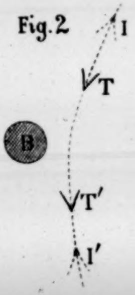


Fig. 2



All ordinary cases belong to the class illustrated by fig. 1. The case of a rigid ring, with cyclic motion (§ 25) established round it as core, belongs to the class illustrated by fig. 2, if the ring be projected through the fluid in the direction perpendicular to its own plane, and contrary to the cyclic motion through its centre.

34. When (§ 66) we substitute vortices for the moving solids, we shall see (§ 67) that the translation is probably always in the direction *with* the impulse. Hence, as illustrated by fig. 1, there is always the deflection, as if by *attraction*, when a group of vortices pass all on one side of a fixed body. This is easily observed, for a simple Helmholtz ring, by sending smoke rings on a large scale, according to

Professor TAIT's plan, in such directions as to pass very near a convex fixed surface. An ordinary 12-inch globe, taken off its bearings and hung by a thin cord, answers very well for the fixed body.

35. The investigation of §§ 30, 31, 32, is clearly applicable to a vortex or a moving body, or of a group of vortices or moving bodies, which keep always near one another (§ 23), passing near a projecting part of the fixed boundary, and being, before and after this collision (§ 29), at a very great distance from every part of the fixed boundary. Thus, a Helmholtz ring projected so as to pass near a projecting angle of two walls, shows a deflection of its course, as if caused by attraction towards the corner.

36. In every case the force-resultant of the impulse is, as we shall presently see (§ 37), determinate when the flow of the liquid across every element of any surface completely enclosing the solids or vortices is given; but not so, from such data, either the axis (§ 6) or the rotational moment, as we see at once by considering the case of a solid sphere (which may afterwards be supposed liquefied) set in motion by a force in any line not through the centre, and a couple in a plane perpendicular to it. For this line will be the "axis," and the impulsive couple will be the rotational moment of the whole motion of the solid and liquid. But the liquid, on all sides, will move exactly as it would if the impulse were merely an impulsive force of equal amount in a parallel line through the centre of the sphere, with therefore this second line for "axis" and zero for rotational moment. For illustration of rotational moment remaining latent in a liquid (with or without solids) until made manifest by actions, tending to alter its axis, or showing effects of centrifugal force due to it; see § 66, and others later.

37. The component impulse in any direction is equal to the corresponding component momentum of the mass enclosed within the surface S, containing all the places of application of the impulse, together with that of the impulsive pressure outwards on this surface. But as the matter enclosed by S (whether all liquid or partly liquid and partly solid) is of uniform density, its momentum will be equal to its mass multiplied into the velocity of the centre of gravity of the space within the surface S supposed to vary so as to enclose always the same matter, and will therefore depend solely on the normal motion of S; that is to say, on the component of the fluid velocity in the direction of the normal at every point of S. And the impulsive fluid pressure, corresponding to the generation of the actual motion from rest, being the time integral of the pressure during the instantaneous generation of the motion, is (§§ 31, 32) equal to $-\phi$, the velocity potential; which (§ 61) is determinate for every point of S, and of the exterior space when the normal component of the fluid motion is given for every point of S. Hence the proposition asserted in § 36. Denoting by $d\sigma$ any element of S; N the normal component of the fluid velocity; a the inclination to OX, of the normal drawn *outwards* through $d\sigma$; and X the x-component of the impulse; we

have for the two parts of this quantity considered above, and its whole value, the following expressions; of which the first is taken in anticipation from § 42—

$$\left.\begin{array}{ll} x\text{-momentum of matter, within S,} & =\iint \mathrm{N}x\,d\sigma \quad (8) \text{ of } \S\, 42 \\ x\text{-component of impulsive pressure on S, outwards,} & =-\iint \varphi \cos \alpha\, d\sigma \end{array}\right\} \quad (1).$$

$$\mathrm{X} = \iint(\mathrm{N}x - \varphi \cos \alpha)\, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

It is worthy of remark that this expression holds for the impulse of all the solids or vortices within S, even if there be others in the immediate neighbourhood outside: and that therefore its value must be zero if there be no solids or vortices within S, and N and ϕ are due solely to those outside.

38. If ϕ be the potential of a magnet or group of magnets, some within S and others outside it, and N the normal component magnetic force, at any point of S, the preceding expression (2) is equal to the x-component of the magnetic moment of all the magnets within S, multiplied by 4π. For let ρ be the density of any continuous distribution of positive and negative matter, having for potential, and normal component force, ϕ and N respectively, at every point of S. We have [THOMSON & TAIT, § 491 (*c*)] $\varrho = -\frac{1}{4\pi}\nabla^2 \varphi$, and therefore

$$\iiint \varrho x\, dx\, dy\, dz = -\frac{1}{4\pi}\iiint x\left(\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2}\right) dx\, dy\, dz \quad . \quad . \quad (3).$$

Now, integrating by parts,* as usual with such expressions, we have

$$\iiint x\frac{d^2\varphi}{dx^2}\,dx\,dy\,dz = \iint x\frac{d\varphi}{dx}\,dy\,dz - \iiint\frac{d\varphi}{dx}\,dx\,dy\,dz = \iint\left(x\frac{d\varphi}{dx} - \varphi\right)dy\,dz\,.$$

Hence, integrating each of the other two terms of (3) once simply, and reducing as usual [THOMSON & TAIT, App. A (*a*)] to a surface integral, we have

$$\iiint \varrho x\, dx\, dy\, dz = -\frac{1}{4\pi}\iint(\mathrm{N}x - \varphi \cos \alpha)\, d\sigma \quad . \quad . \quad . \quad . \quad (4);$$

which proves the proposition, and also, of course, that if there be no matter within S, the value of the second member is zero.

39. Hence, considering the magnetic and hydrokinetic analogous systems with the sole condition that at every point of some particular closed surface, the magnetic potential is equal to the velocity potential, we conclude that 4π times the magnetic moment of all the magnetism within any surface, in the magnetic system, is equal to the force-resultant of the impulse of the solids or vortices within the corresponding surface in the hydrokinetic system; and that the directions of the magnetic axis and of the force-resultant of the impulse are the same. For the theory of magnetism, it is interesting to remark that indeterminate distributions of magnetism within the solids, or portions of fluid to which initiating

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* The process here described leads merely to the equation obtained by taking the last two equal members of App. A (1) (THOMSON & TAIT) for the case $\alpha = 1$, $\mathrm{U} = \varphi$, $\mathrm{U}' = x$.

forces (§ 3) were applied, or determinate distributions in infinitely thin layers at their surfaces, may be found, which through all the space external to them shall produce the same potential as the velocity-potential, and therefore the same distribution of force as the distribution of velocity through the whole fluid. But inasmuch as when the magnetic force in the interior of a magnet is defined in the manner explained in § 48 (2) of my "Mathematical Theory of Magnetism,"* it is expressible through all space by the differential coefficients of a potential; and, on the contrary, for the kinetic system $u\,dx + v\,dy + w\,dz$ is not a complete differential generally through the spaces occupied by the solids, the agreement between resultant force and resultant flow holds only through the space exterior to the magnets and solids in the magnetic and kinetic systems respectively. But if the other definition of resultant force within a magnet, ["Math. Theory of Magnetism," § 77, foot-note, and § 78], published in preparation for a 6th chapter "On Electro-magnets" (still in my hands in manuscript, not quite completed). and which alone can be adopted for spaces occupied by non-magnetic matter traversed by electric currents, the magnetic force has not a potential within such spaces; and we shall see (§ 68) that determinate distributions of closed electric currents through spaces corresponding to the solids of the hydrokinetic system can be found which shall give for every point of space, whether traversed by electric currents or not, a resultant magnetic force, agreeing in magnitude and direction with the velocity, whether of solid or fluid, at the corresponding point of the hydrokinetic system. This thorough agreement for all space renders the electro-magnetic analogue preferable to the magnetic; and, having begun with the magnetic analogous system only because of its convenience for the demonstration of § 38, we shall henceforth chiefly use the purely electro-magnetic analogue.

40. To prove the formula used in anticipation, in § 37 (1) we must now (§§ 41, 42, 43) find the momentum of the whole matter—fluid, fluid and solid, or even solid alone—at any instant within a closed surface S, in terms of the normal component velocity of the matter at any point of this surface, or, which is the same, the normal velocity of this surface itself, if we suppose it to vary so as to enclose always the same matter.

41. Let V be the volume of the space bounded by any varying closed surface S. As yet we need not suppose V constant. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of of the centre of gravity. We have

$$V\bar{x} = \tfrac{1}{2}\iint[x^2\,dy\,dz] \quad . \quad . \quad . \quad . \quad (5),$$

where [ ] indicates that the expression within it is to be taken between proper limits for S. Now as S varies with the time, the area through which $\iint dy\,dz$ is taken will in general vary; but the increments or decrements which it experiences at different parts of the boundary of this area, in the infinitely small time dt, contribute no increments or decrements to $\iint[x^2 dy\, dz]$, as we see most easily by first supposing S to be a surface everywhere convex outwards. Hence

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* Trans. R. S. Lond., 1851; or "THOMSON's Electrical Papers." Macmillan. 1869.

$$\frac{d}{dt}\iint[x^2 dy\, dz] = \iint\left[\frac{d\,(x^2)}{dt}\, dy\, dz\right] = 2\iint[x\,\frac{dx}{dt}\, dy\, dz] \quad . \qquad (6).$$

But if N denote the velocity with which the surface moves in the direction of its outward normal at (x, y, z), we have, in the preceding expression

$$\frac{dx}{dt} = \text{N}\sec\alpha \quad . \quad . \quad . \qquad (7),$$

if α be the inclination of the outward normal to OX. Hence

$$\frac{d(\text{V}\bar{x})}{dt} = \iint[x\text{N}\sec\alpha\, dy\, dz].$$

But the condition as to limits indicated by [ ] are clearly satisfied, if, $d\sigma$ denoting an element of the surface, such that

$$dy\, dz = \cos\alpha\, d\sigma,$$

we simply take $\iint d\sigma$ over the whole surface. Thus we have

$$\frac{d(\text{V}\bar{x})}{dt} = \iint x\text{N}\, d\sigma \quad . \quad . \quad . \qquad (7);$$

42. In any case in which V is constant, this becomes

$$\text{V}\,\frac{d\bar{x}}{dt} = \iint x\,\text{N}\, d\sigma \quad . \quad . \quad . \quad . \qquad (8).$$

If now the varying surface, S, is the boundary of a portion of the matter—fluid or solid—of uniform density unity, with whose motions we are occupied, the x-component momentum of this portion is $\text{V}\frac{dx}{dt}$; and, therefore, equation (8) is the required (§ 40) expression.

43. The same formulæ (7) and (8) are proved more shortly of course by the regular analytical process given by POISSON* and GREEN† in dealing with such subjects; thus, in short. Let u, v, w be the components of velocity, of any matter, compressible or incompressible, at any point (x, y, z) within S; and let c denote the value at this point of $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$, so that

$$\frac{du}{dx} = c - \left(\frac{dv}{dy} + \frac{dw}{dz}\right) \quad . \quad . \quad . \quad . \qquad (9).$$

We have, for the component momentum of the whole matter within S, if of unit density at the instant considered,

$$\iiint u\, dx\, dy\, dz = \iint ux\, dy\, dz - \iiint x\,\frac{du}{dx}\, dx\, dy\, dz \quad . \qquad (10).$$

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[*] Théorie de la Chaleur, § 60.

[†] Essay on Electricity and Magnetism.

But by (9)

$$\iiint x \frac{du}{dx}\, dx\, dy\, dz = \iiint cx\, dx\, dy\, dz - \iiint x\left(\frac{dv}{dy} + \frac{dw}{dz}\right) dx\, dy\, dz$$

and by simple integrations,

$$\iiint x\left(\frac{dv}{dy} + \frac{dw}{dz}\right) dx\, dy\, dz = \iint x(v\, dx\, dz + w\, dx\, dy).$$

Using these in (10), and altering the expression to a surface integral, as in THOMSON & TAIT, App. A (*a*), we have

$$\begin{aligned} \textstyle\iiint u\, dx\, dy\, dz &= \textstyle\iint x\,(u\, dy\, dz + v\, dz\, dx + w\, dx\, dy) - \iiint cx\, dx\, dy\, dz \\ &= \textstyle\iint x\, \mathrm{N}\, d\sigma - \iiint cx\, dx\, dy\, dz \quad . \quad . \quad . \quad . \quad . \qquad (11), \end{aligned}$$

which clearly agrees with (7).

When this mass is incompressible, we have $c = o$ by the formula so ill named the equation "of continuity" (THOMSON & TAIT, § 191), and we fall upon (8.)

The proper analytical interpretation of the differential coefficients $\frac{du}{dx}$, &c., and of the equation of continuity, when, as at the surfaces of separation of fluid and solids, u, v, w are discontinuous functions, having abruptly varying values, presents no difficulty.

44. In the theory of the impulse applied to the collision (§ 29) of solids or vortices moving through a liquid, the force-resultant of the impulse corresponds, as we have seen, precisely to the resultant momentum of a solid in the ordinary theory of impact. Some difficulty may be felt in understanding how the zero-momentum (§ 4) of the whole mass is composed; there being clearly positive momentum of solids and fluids in the direction of the impulse in some localities near the place of its application, and negative in others. [Consider, for example, the simple case of a solid of revolution struck by a single impulse in the line of its axis. The fluid moves in the direction of the impulse, before and behind the body, but in the contrary direction in the space round its middle.] Three modes of dividing the whole moving mass present themselves as illustrative of the distribution of momentum through it; and the following propositions (§ 45) with reference to them are readily proved (§§ 46, 47, 48).

45. I. Imagine any cylinder of finite periphery, not necessarily circular, completely surrounding the vortices (or moving solids), and any other surrounding none, and consider the infinitely long prisms of variously moving matter at any instant surrounded by these two cylinders. The component momentum parallel to the length of the first is equal to the component of the impulse parallel to the same direction; and that of the second is zero.

II. Imagine any two finite spherical surfaces, one enclosing all the vortices or moving solids, and the other none. The resultant-momentum of the whole matter enclosed by the first is in the direction of the impulse, and is equal to $\frac{2}{3}$ of its value. The resultant-momentum of the whole fluid enclosed by the second is the same as if it all moved with the same velocity, and in the same direction, as at its centre.

III. Imagine any two infinite planes at a finite distance from one another and from the field of motion, but neither cutting any solid or vortex. The component perpendicular to them of the momentum of the matter occupying at any instant the space between them (whether this includes none, some, or all of the vortices or moving solids) is zero.

46. To prove these propositions:—

I. Consider in either case a finite length of the prism extending to a very great distance in each direction from the field of motion, and terminated by plane or curved ends. Then, the motion being, as we may suppose (§ 61) started from rest by impulsive pressures on the solids [or (§ 66) on the portions of fluid constituting the vortices]; the impulsive fluid pressure on the cylindrical surface can generate no momentum parallel to the length; and to generate momentum in this direction there will be, in case 1, the impressed impulsive forces on the solids, and the impulsive fluid pressures on the ends; but in case 2 there will be only the impulsive fluid pressure on the ends. Now, the impulsive fluid pressures on the ends diminish [§ 50 (15)] according to the inverse square of the distance from the field of motion, when the prism is prolonged in each direction, and are therefore infinitely small when the prisms are infinitely long each way. Whence the proposition I.

47. By using the harmonic expansions § 19, (14), (15), in the several expressions (1), (2), of § 37, (1), (2); and the fundamental theorem

$$\iint \mathfrak{S}_i \mathfrak{S}_{i'} \, d\sigma = 0 ,$$

of the harmonic analysis [THOMSON & TAIT, App. B. (16)]; and putting $S_i = 0$ for one case, and $T_i = 0$ for the other; we prove the two parts of Prop. II., § 45 immediately.

48. To prove Prop. II., § 45, the well-known theory of electric images in a plane conductor* may be conveniently referred to. It shows that if N_1 denotes the normal component force at any point of an infinite plane due to any distribution, μ, of matter in the space lying on one side of the plane, a distribution of matter over the plane having $\frac{1}{2\pi} N_1$ for surface density at each point exerts the same force as μ through all the space on the other side of the plane, and therefore that the whole quantity of matter in that surface distribution is equal to the

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* THOMSON, Camb. and Dub. Math. Journal, 1849; LIOUVILLE'S Journal, 1845 and 1847; or Reprints of Electrical Papers, (Macmillan, 1869.)

whole quantity of matter in μ.* Hence, $\iint d\sigma$, denoting integration over the infinite *plane*

$$\iint N_1 \, d\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

if the whole quantity of matter in μ be zero. Hence, if N be the normal force due to matter through space on both sides of the plane, provided the whole quantity of matter on each side separately is zero,

$$\iint N \, d\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14);$$

since N is the sum of two parts, for each of which separately (12) holds. This translated into hydrokinetics, shows that the whole flow of matter across any infinite plane is zero at every instant when it cuts no solids or vortices. Hence, and from the uniformity of density which (§ 3), we assume, the centre of gravity of the matter between any two infinite fixed parallel planes, has no motion in the direction perpendicular to them at any time when no vortex or moving solid is cut by either: which is Prop. III. of § 4 in other words.

49. The integral flow of matter across any surface whatever, imagined to divide the whole volume of the finite fixed containing vessel of § 1 into two parts is necessarily zero, because of the uniformity of density; and therefore the momentum of all the matter bounded by two parallel planes, extending to the inner surface of the containing vessel, and the portion of this surface intercepted between them has always zero for its component perpendicular to these planes, whether or not moving solids or vortices are cut by either or both these planes. But it is remarkable that when any moving solid or vortex is cut by a plane, the integral flow of matter across this plane (if the containing vessel is infinitely distant on all sides from the field of motion), converges to a generally *finite* value, as the plane is extended to very great distances all round from the field of motion, which are still infinitely small in comparison with the distances to the containing vessel; and diminishes from that finite value to zero by another convergence, when the distances to which the plane is extended all round begin to be comparable with, and ultimately become equal to, the distances of the curve in which it cuts the containing vessel. Hence we see how it is that the condition of neither plane cutting any moving solid or vortex is necessary to allow § 46. III. to be stated without reference to the containing vessel, and are reminded that

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* This is verified synthetically with ease, by direct integrations showing (whether by Cartesian or polar plane co-ordinates), that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{a\,dy\,dz}{(a^2+y^2+z^2)^{\frac{3}{2}}} = 2\pi \quad . \quad . \quad . \quad . \quad (13).$$

And taking $\frac{d}{da}$ of this, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(y^2+z^2-2a^2)\,dy\,dz}{(a^2+y^2+z^2)^{\frac{5}{2}}} = 0 \quad . \quad . \quad . \quad . \quad (12)'.$$

the synthesis of (12).

the equality to zero asserted in this proposition is proved in § 48 to be approximated to when the planes are extended to distances all round, which, though infinitely short of the distances to the containing vessel, are very great in comparison with their perpendicular distances from the most distant parts of the field of motion.

50. The convergencies concerned in § 46, I., III. may be analysed thus. Perpendicular to the resultant impulse draw any two planes on the two sides of the field of motion, with all the moving solids and vortices between them, and divide a portion of the space between them into finite prismatic portions by cylindrical (or plane) surfaces perpendicular to them. Suppose now one of these prismatic portions to include all the moving solids and vortices, and without altering the prismatic boundary, let the parallel planes be removed in opposite directions to distances each infinite (or very great) in comparison with distance of the most distant of the moving solids or vortices. By § 46, I., the momentum of the motion within this prismatic space is (approximately) equal to the force-resultant, I, of the impulse, and that of the motion within any one of the others is (approximately) zero.

But the sum of these (approximately) zero values must, on account of § 46, III., be equal to — I, if the portions of the planes containing the ends of the prismatic spaces be extended to distances very great in comparison with the distance between the planes. To understand this, we have only to remark that if ϕ denotes the velocity potential at a point distant D from the middle of the field, and x from a plane through the middle perpendicular to the impulse, we have (§ 53) approximately,

$$\phi = -\frac{\mathrm{I}\,x}{4\pi\,\mathrm{D}^3} \quad . \quad . \quad . \quad . \quad . \quad (15),$$

provided D be great in comparison with the radius of the smallest sphere enclosing all the moving solids or vortices. Hence, putting $x = \pm a$ for the two planes under consideration, denoting by A the area of either end of one of the prismatic portions, and calling D *the proper mean distance* for this area, we have (§ 46) for the momentum of the fluid motion within this prismatic space, provided it contains no moving solids or vortices,

$$-2\frac{\mathrm{I}\,a}{4\pi\,\mathrm{D}^3}\mathrm{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

This vanishes when $\frac{\mathrm{A}}{\mathrm{D}^2}$ is an infinitely small fraction (as $\frac{a}{\mathrm{D}}$ is at most unity); but it is finite if $\frac{\mathrm{A}}{\mathrm{D}^2}$ is finite, provided $\frac{a}{\mathrm{D}}$ be not infinitely small. And its integral value (compare § 48, footnote) converges to — I, when the portion of area included in the integration is extended till $\frac{a}{\mathrm{D}}$ is infinitely small for all points of its boundary.

51. Both as regards the mathematical theory of the convergence of definite integrals, and as illustrating the distribution of momentum in a fluid, it is interesting to remark that, u denoting component velocity parallel to x, at any point (x, y, z), the integral $\iiint u\,dx\,dy\,dz$, expressing momentum, may, as is readily proved, have any value from $-\infty$ to $+\infty$ according to the portions of space through which it is taken.

52. As a last illustration of the distribution of momentum, let the containing vessel be spherical of finite radius a.

We have, as in § 19,

$$\left.\begin{aligned}\varphi = \mathrm{S}_0 + \mathrm{S}_1 r + \mathrm{S}_2 r^2 + \&c.,\\ + \mathrm{T}_1 r^{-2} + \mathrm{T}_2 r^{-3} + \&c.,\end{aligned}\right\} \quad . \quad . \quad . \quad (14),$$

each series converging, provided r is less than a, and greater than the radius of the smallest concentric spherical surface enclosing all the solids or vortices. Now, by the condition that there be no flow across the fixed containing surface, we must have

$$\frac{d\varphi}{dr} = 0, \text{ when } r = a \quad . \quad . \quad . \quad . \quad . \quad (15),$$

which gives

$$\mathrm{S}_i = \frac{i+1}{i}\,\frac{\mathrm{T}_i}{a^{2i+1}} \quad . \quad . \quad . \quad . \quad . \quad (16);$$

and (14) becomes

$$\varphi = \frac{\mathrm{T}_1}{r^2}\left(1 + 2\frac{r^3}{a^3}\right) + \frac{\mathrm{T}_2}{r^3}\left(1 + \frac{3}{2}\frac{r^5}{a^5}\right) + \&c. \quad . \quad . \quad (17).$$

But [§ 37 (1)] if the whole amount of the x-component of impulsive pressure exerted by the fluid within the spherical surface of radius r, upon the fluid round it be denoted by F, we have

$$\mathrm{F} = -\iint \varphi \cos\theta\, d\sigma \quad . \quad . \quad . \quad (18),$$

θ being the inclination to OX of the radius through $d\sigma$. Now $\cos\theta$ is a surface harmonic of the first order, and therefore all the terms of the harmonic expansion, except the first, disappear in the integral, which consequently becomes

$$\mathrm{F} = -\left(1 + 2\frac{r^3}{a^3}\right)\iint \mathrm{T}_1 \cos\theta\, \frac{d\sigma}{r^2} \quad . \quad . \quad . \quad (19).$$

Now let

$$\mathrm{T}_1 = -\frac{\mathrm{A}x + \mathrm{B}y + \mathrm{C}z}{r} \quad . \quad . \quad . \quad . \quad (20),$$

this being [THOMSON & TAIT, App. B, §§ i, j] the most general expression for a surface harmonic of the first order. We have $\cos\theta = \frac{x}{r}$: and therefore (by spherical harmonics, or by the elementary analysis of moments of inertia of a uniform spherical surface),

$$-\iint \mathrm{T}_1 \cos\theta \frac{d\sigma}{r^2} = \frac{\mathrm{A}}{r^4}\iint x^2\, d\sigma = \frac{4\pi\mathrm{A}}{3} \quad . \quad . \quad (21);$$

and (19) becomes

$$\mathrm{F} = \left(1 + 2\frac{r^3}{a^3}\right) . \frac{4\pi\mathrm{A}}{3} \quad . \quad . \quad . \quad . \quad (22):$$

Whence, if X denote the x-momentum of the fluid at any instant in the space between concentric spherical surfaces of radius r and r',

$$\mathrm{X} = -\frac{2}{3}\frac{r^3 - r'^3}{a^3} 4\pi\mathrm{A} \quad . \quad . \quad . \quad . \quad (23).$$

If r and r' be each infinitely small in comparison with a, this expression vanishes, as it ought to do, in accordance with § 45, II. But if

$$\left.\begin{aligned} &\frac{r}{a} = 0\,, \quad \& \; r = a\,, \\ &\mathrm{X} = -\tfrac{2}{3} . \, 4\pi\mathrm{A} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (24),$$

it becomes

fulfilling § 4, by showing in the fluid outside the spherical surface of radius r' a momentum equal and opposite to that (§ 45, II.) of the whole matter, whether fluid or solid, within that surface.

53. Comparing § 47 and § 52, we see that if X, Y, Z be rectangular components of the force-resultant of the impulse, the term $\mathrm{T}_1\, r^{-2}$ of the harmonic expansion (14) is as follows:—

$$\mathrm{T}_1\, r^{-2} = \frac{\mathrm{X}x + \mathrm{Y}y + \mathrm{Z}z}{4\pi r^3} \quad . \quad . \quad . \quad (25),$$

provided all the solids and vortices taken into account are within a spherical surface whose radius is very small in comparison with the distances of all other vortices or moving solids, and with the shortest distance to the fixed bounding surface.

54. Helmholtz, in his splendid paper on Vortex Motion, has made the very important remark, that a certain fundamental theorem of Green's, which has been used to demonstrate the determinateness of solutions in hydrokinetics, is subject to exception when the functions involved have multiple values. This calls for a serious correction and extension of elementary hydrokinetic theory, to which I now proceed.

55. In the general theorem (1) of Thomson & Tait, App. A let $a = 1$. It becomes

$$\iiint\left(\frac{d\phi}{dx}\frac{d\phi'}{dx} + \frac{d\phi}{dy}\frac{d\phi'}{dy} + \frac{d\phi}{dz}\frac{d\phi'}{dz}\right) dx\,dy\,dz = \iint d\sigma\,\phi\,\eth\phi' - \iiint dx\,dy\,dz\,\phi\nabla^2\phi'$$

$$= \iint d\sigma\,\phi'\,\eth\phi - \iiint dx\,dy\,dz\,\phi'\nabla^2\phi \quad . \quad . \quad . \quad . \quad (1),$$

which is true without exception if ϕ and ϕ' denote any two *single-valued* functions of x, y, z; $\iiint dx\,dy\,dz$ integration through the space enclosed by any finite closed surface, S; $\iint d\sigma$ integration over the area of this surface; and $\mathfrak{d}$ rate of variation per unit of length in the normal direction at any point of it. This is GREEN'S original theorem, with HELMHOLTZ'S limitation added (in italics.) The reader may verify it for himself.

56. But if either ϕ or ϕ' is a many-valued function, and the differential coefficients $\frac{d\varphi}{dx}, \ldots, \frac{d\varphi'}{dx}, \ldots,$ each single-valued, the double equation (1) cannot be generally true. Its first member is essentially unambiguous; but the process of integration by which the second member or the third member is found, would introduce ambiguity if ϕ or if ϕ' is many-valued. In one case the first member, though not equal to the ambiguous second, would be equal to the third, provided ϕ' is not also many-valued; and in the other, the first member, though not equal to the third, would be equal to the second, provided ϕ is not many-valued.

For example, let

$$\varphi' = \tan^{-1}\frac{y}{x} \qquad . \qquad . \qquad . \qquad . \qquad (2).$$

and let S consist of the portions of two planes perpendicular to OZ, intercepted between two circular cylinders having OZ for axis, and the portions of these cylinders intercepted between the two planes. The inner cylindrical boundary excludes from the space bounded by S, the line OZ where ϕ' has an infinite number of values, and $\frac{d\varphi'}{dx}$, and $\frac{d\varphi'}{dz}$ have infinite values. We have

$$\frac{d\varphi'}{dx} = \frac{-y}{x^2+y^2},\ \frac{d\varphi'}{dy} = \frac{x}{x^2+y^2} \qquad . \qquad . \qquad . \qquad (3),$$

and at every point of S, $d\phi' = 0$. Then, if ϕ be single-valued, there is no failure in the process proving the equality between the first and second members of (1), which becomes

$$\iiint \frac{x\frac{d\varphi}{dy} - y\frac{d\varphi}{dx}}{x^2+y^2}\,dx\,dy\,dz = 0 \qquad . \qquad . \qquad . \qquad (4).$$

Compare § 14 (6) to end.

The third member of (1) becomes

$$\iint d\sigma \tan^{-1}\frac{y}{x}\,\mathfrak{d}\varphi - \iiint \tan^{-1}\frac{y}{x}\,\nabla^2\varphi\,dx\,dy\,dz \qquad . \qquad . \qquad (5),$$

which is no result of unambiguous integration of the first member through the space enclosed by S, as we see by examining, in this case, the particular meaning of each step of the ordinary process in rectangular co-ordinates for proving GREEN'S theorem. It is thus seen that we must add to (5) a term

$$2\pi\iint dx\,dz\left(\frac{d\varphi}{dy}\right)_{y=0},$$

if in its other terms the value of $\tan^{-1}\frac{y}{x}$ is reckoned continuously round from one side of the plane ZOX to the other: or

$$-2\pi\iint dy\,dz\left(\frac{d\phi}{dx}\right)_{x=0},$$

if the continuity be from one side of ZOY to the other; to render it really equal to the first member of (1). Thus, taking for example the first form of the added term, we now have for the corrected double equation (1) for the case of $\phi' = \tan^{-1}\frac{y}{x}$, ϕ any single valued function, and S the surface, composed of the two co-axal cylinders and two parallel planes specified above:

$$\iiint \frac{x\dfrac{d\phi}{dz} - y\dfrac{d\phi}{dx}}{x^2+y^2}\,dx\,dy\,dz = 0 = 2\pi\iint dx\,dz\left(\frac{d\phi}{dy}\right)_{y=0} + \iint d\sigma \tan^{-1}\frac{y}{x}\mathfrak{d}\phi$$

$$-\iiint dx\,dy\,dz \tan^{-1}\frac{y}{x}\nabla^2\phi \quad . \quad . \quad . \quad (6).$$

But if we annex to S any barrier stopping circulation round the inner cylindrical core, all ambiguity becomes impossible, and the double equation (1) holds. For instance, if the barrier be the portion of the plane ZOX, intercepted between the co-axal cylinders and parallel planes constituting the S of § 55, so that $\iint d\sigma$ must now include integration over each side of this rectangular area; (6) becomes simply the strict application of (1) to the case in question.

57. The difficulty of the exceptional interpretation of GREEN's theorem for the class of cases exemplified in §§ 55 and 56, depends on the fact that $\int F ds$ may have different values when reckoned along the lengths of different curves, drawn within the space bounded by S, from a point P to a point Q; ds being an infinitesimal element of the curve, and F the rate of variation of ϕ per unit of length along it. Let PCQ, PC′Q be two curves for which the $\int F ds$ has different values; and let both lie wholly within S. If we draw any curve from P to Q; make it first coincide with PCQ, and then vary it gradually until it coincides with PC′Q; it must in some of its intermediate forms cut the bounding surface S: for we have

$$F ds = \frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{dz}dz$$

throughout the space contained within S, and $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, are each of them unambiguous by hypothesis; which implies that $\int F ds$ has equal values for all gradual variations of one curve between P and Q, each lying wholly within S. Now, in a simply continuous space, a curve joining the points P and Q may be gradually varied from any curve PCQ to any other PC′Q, and therefore if the space contained within S be simply continuous, the difficulty depending on the multiplicity of value of ϕ or ϕ' cannot exist. And however multiply continuous (§ 58) the space may be, the difficulty may be evaded if we annex to S a surface or surfaces stopping every aperture or passage on the openness of which its multiple continuity depends; for these annexed surfaces, as each of them occupies no space, do not disturb the triple integrations (1), and will, therefore, not alter the values of its first member; but by removing the multiplicity of continuity, they free each of the integrations by parts, by which its second or third members are obtained, from all ambiguity. To avoid circumlocution, we shall call β the addition thus made to S; and further, when the space within S is (§ 58) not merely doubly but triply, or quadruply, or more multiply, continuous, we shall designate by β_1, β_2; or β_1, β_2, β_3; and so on; the several parts of β required in any case to stop all multiple continuity of the space. These parts of β may be quite detached from one another, as when the multiple continuity is that due to detached rings, or separate single tunnels in a solid. But one part β_1 may cut through part of another, β_2, as when two rings (§ 58, diagram) linked into one another without touching constitute part of the boundary of the space considered. And we shall denote by $\iint d\varsigma$, integration over the surface β, or over any one of its parts, β_1, β_2, &c. Let now P and Q be each infinitely near a point B, of β, but on the two sides of this surface. Let κ denote the value of $\int F ds$ along any curve lying wholly in the space bounded by S, and joining PQ without cutting the barrier; this value being the same for all such curves, and for all positions of B to which it may be brought without leaving β, and without making either P or Q pass through any part of β. That is to say, κ is a single constant when the space is not more than doubly continuous; but it denotes one or other of n constants $\kappa_1, \kappa_2, \ldots \kappa_n$, which may be all different from one another, when the space is n-ply continuous. Lastly, let κ' denote the same element, relatively to ϕ', as κ relatively to ϕ. We find that the first steps of the integrations by parts now introduce, without ambiguity, the additions

$$\Sigma\kappa \iint d\varsigma\, \mathfrak{d}\varphi', \text{ and } \Sigma\kappa' \iint d\varsigma\, \mathfrak{d}\varphi \quad . \quad . \quad . \quad (6),$$

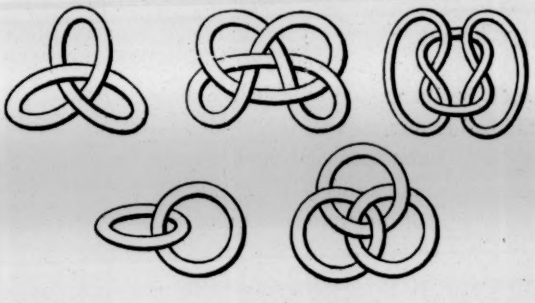
to the second and third numbers of (1): Σ denoting summation of the integrations for the different constituents $\beta_1, \beta_2, \ldots$ of β; but only a single term when the space is (§ 58) not more than doubly continuous. GREEN's theorem thus corrected becomes

$$\iint\left(\frac{d\varphi}{dx}\frac{d\varphi'}{dx} + \frac{d\varphi}{dy}\frac{d\varphi'}{dy} + \frac{d\varphi}{dz}\frac{d\varphi'}{dz}\right) dx\, dy\, dz = \iint d\sigma\, \varphi\mathfrak{d}\varphi' + \Sigma\kappa \iint d\varsigma\, \mathfrak{d}\varphi' - \iiint \varphi \nabla^2 \varphi'\, dx\, dy\, dz$$

$$= \iint d\sigma\, \varphi'\mathfrak{d}\varphi + \Sigma\kappa' \iint d\varsigma\, \mathfrak{d}\varphi - \iiint \varphi' \nabla^2 \varphi\, dx\, dy\, dz \quad . \quad (7).$$

58. Adopting the terminology of RIEMANN, as known to me through HELMHOLTZ, I shall call a finite position of space n-ply continuous when its bounding surface is such that there are n irreconcilable paths between any two points in it. To prevent any misunderstanding, I add (1), that by *a* portion of space I mean such a portion that any point of it may be travelled to from any other point of it, without cutting the bounding surface; (2), that the "paths" spoken of all lie within the portion of space referred to; and (3), that by irreconcilable paths between two points P and Q; I mean paths such, that a line drawn first along one of them cannot be gradually changed till it coincides with the other, being always kept passing through P and Q, and always wholly within the portion of space considered. Thus, when all the paths between any two points are reconcilable, the space is simply continuous. When there are just two sets of paths, so that each of one set is irreconcilable with any one of the other set, the space is doubly continuous; when there are three such sets it is triply continuous, and so on. To avoid circumlocutions, we shall suppose S to be the boundary of a hollow space in the interior of a solid mass, so thick that no operations which we shall consider shall ever make an opening to the space outside it. A tunnel through this solid opening at each end into the interior space constitutes the whole space doubly continuous; and if more tunnels be made, every new one adds one to the degree of multiple continuity. When one such tunnel has been made, the surface of the tunnel is continuous with the whole bounding surface of the space considered; and in reckoning degrees of continuity, it is of no consequence whether the ends of any fresh tunnel be in one part or another of this whole surface. Thus, if two tunnels be made side by side, a hole anywhere opening from one of them into the other adds one to the degree of multiple continuity. Any solid detached from the outer bounding solid, and left, whether fixed or movable in the interior space, adds to the bounding surface an isolated portion, but does not interfere with the reckoning of multiple continuity. Thus, if we begin with a simply continuous space bounded outside by the inner surface of the supposed external solid, and internally by the boundary of the detached solid in its interior, and if we drill a hole in this solid we produce double continuity. Two holes, or two solids in the interior each with one hole (such as two ordinary solid rings), constitute triple continuity, and so on. A sponge-like solid whose pores communicate with one another, illustrates a high degree of multiple continuity, and it is of no consequence whether it is attached to the external bounding solid or is an isolated solid in the interior. Another type of multiple continuity, that presented by two rings linked in one another, was referred to in § 57.

When many rings are linked into one another in various combinations, there are complicated mutual intersections of the several partial barriers $\beta_1, \beta_2, \ldots$ required to stop all multiple continuity. But without having any portion of the bounding solid detached, as in that case in which one at least of the two rings is loose, we have varieties of multiple continuity curiously different from that illustrated by a single ordinary straight or bent tunnel, illustrated sufficiently by the simplest types, which are obtained by boring a tunnel along a line agreeing in form with the axis of a cord or wire on which a simple knot is tied; and by fixing the two ends of wire with a knot on it to the bounding solid, so that the surface of the wire shall become part of the bounding surface of the space considered, the knot not being pulled tight, and the wire being arranged not to touch itself in any point; or by placing a knotted wire, with its ends united, in the interior of the space. No amount of knotting or knitting, however complex, in the cord whose axis indicates the line of tunnel, complicates in any way the continuity of the space considered, or alters the simplicity of the barrier surface required to stop the circulation. But it is otherwise when a knotted or knitted wire forms part of the bounding solid. A single simple knot, though giving only double continuity, requires a curiously self-cutting surface for stopping barrier: which, in its form of minimum area, is beautifully shown by the liquid film adhering to an endless wire, like the first figure, dipped in a soap solution and removed. But no complication of these types, or of combinations of them with one another, eludes the statements and formulæ of § 57.



59. I shall now give a dynamical lemma, for the immediate object of preparing to apply GREEN's corrected theorem (§ 57) to the motion of a liquid through a multiply continuous space. But later we shall be led by it to very simple demonstrations of HELMHOLTZ's fundamental theorems of vortex motion; and shall see that it may be used as a substitute for the common equations of hydrokinetics.

(Lemma). An endless finite tube* of infinitesimal normal section, being given full of liquid (whether circulating round through it, or at rest) is altered in shape,

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* A finite length of tube with its ends done away by uniting them together.

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